Tripartite entanglement in a noninertial frame

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Tripartite entanglement is examined when one of the three parties moves with a uniform acceleration with respect to other parties. As the Unruh effect indicates, tripartite entanglement exhibits a decreasing behavior with increasing acceleration. Unlike bipartite entanglement, however, tripartite entanglement does not completely vanish in the infinite acceleration limit. If the three parties, for example, share the Greenberger-Horne-Zeilinger or W state initially, the corresponding π-tangle, one of the measures of tripartite entanglement, is shown to be $\pi/6 \sim 0.524$ or 0.176 in this limit, respectively. This fact indicates that tripartite quantum-information processing may be possible even if one of the parties approaches the Rindler horizon. The physical implications of this striking result are discussed in the context of black-hole physics.

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I. INTRODUCTION

It is well known that entanglement of quantum states is a genuine physical resource for various quantum-information tasks such as quantum teleportation [1], quantum cryptography [2], and quantum computer technology [3]. For this reason, much attention has recently been paid to the various properties of entanglement [4].

In addition to the pure quantum-mechanical aspect, it is also important to analyze the entanglement of the multipartite quantum state in the relativistic framework. Evidently, this is an interesting subject from a fundamental point of view. Furthermore, it is also important from a practical perspective, because many modern experiments on quantum-information processing use photons or other particles which have relativistic velocities. The bipartite entanglement between inertial frames was investigated in Ref. [5]. The remarkable fact regarding entanglement between inertial frames is its conservation, although the entanglement between some degrees of freedom can be transferred to others. Still, it is not obvious why entanglement between inertial frames is preserved.

The bipartite entanglement between noninertial frames was initially studied by Fuentes-Schuller and Mann (FM) in Ref. [6]. They showed that the maximal bipartite entanglement between inertial parties is degraded if the observers are relatively accelerated. With increasing acceleration, degradation of entanglement becomes larger and larger, and eventually the bipartite state reduces to the separable state at infinite acceleration. This phenomenon is sometimes called Unruh decoherence and is closely related to the Unruh effect [7]. Due to the resemblance between the Unruh effect and Hawking radiation [8], FM predicted that the degradation of entanglement occurs in black-hole physics. The degradation phenomenon of bipartite entanglement in a Schwarzschild black hole was investigated in Ref. [9]. Although entanglement is degraded near a Schwarzschild black hole as FM predicted, there is a subtle difference arising due to the difference between an event horizon in Schwarzschild space-time and an acceleration horizon in Rindler space-time. Recently, quantum teleportation between noninertial observers has also been discussed in detail in Ref. [10].

Before we start discussing our main subject, it is worthwhile noting that a few years ago there was a debate on the physical relevance of the Unruh effect [11]. Authors in Ref. [11] argue that the Unruh effect can be realized only when the quantum field operator satisfies a particular boundary condition at the spatial boundary of the space-time manifold. They argued that due to this boundary condition, the Unruh effect can be realized in a double Rindler wedge rather than in the usual Minkowski space. Subsequently, refutation and defense of their argument were published in Ref. [12]. However, a discussion on this debate in detail is beyond the scope of present paper. Here, we will discuss tripartite entanglement in a noninertial frame assuming that the Unruh effect is physically relevant in the Minkowski space.

In this paper, we discuss tripartite entanglement in a noninertial frame. As far as we know there are two entanglement measures which quantify the genuine tripartite entanglement: three-tangle [13] and π-tangle [14]. The three-tangle has many nice properties and exactly coincides with the modulus of Cayley's hyperdeterminant [15]. It is also an invariant quantity under the local $SL(2,C)$ transformation [16]. Despite its nice features, it has a drawback due to its calculational difficulty. Since we need an optimal decomposition for the analytical computation of the three-tangle for a given tripartite mixed state, it is highly difficult to compute the three-tangle analytically except in a few rare cases [17]. In order to escape this difficulty, we adopt the π-tangle for the quantification of tripartite entanglement solely because of its calculational easiness. The physical roles of the three-tangle and π-tangle in real quantum-information processing was recently discussed in Ref. [18] in detail.

In this paper we are considering the following situation. Let Alice, Bob, and Charlie share the Greenberger-Horne-Zeilinger (GHZ) [19] or W state [20] initially when they are not moving relatively. Subsequently, Charlie moves with a uniform acceleration with respect to Alice and Bob. We compute the π-tangle as a function of Charlie's acceleration. It is shown in this paper that the π-tangle, in general, decreases with increasing acceleration as in bipartite entanglement. However, we show that unlike bipartite entanglement, the π-tangle does not completely vanish even if Charlie moves with an infinite acceleration. This is a striking result in the sense that this fact implies the possibility of tripartite quantum-information processing although Charlie approaches the Rindler horizon.

This paper is organized as follows. In Sec. II we consider a situation where Alice, Bob, and Charlie share the GHZ state initially. It is shown that the resulting π-tangle decreases when Charlie's acceleration increases from 1 at zero acceleration to $\pi/6 \sim 0.524$ at infinite acceleration. In Sec. III the GHZ state in the previous section is replaced with the W state. It is shown that the π-tangle in this case also decreases when acceleration increases from $4(\sqrt{5}-1)/9 \sim 0.55$ at zero acceleration to 0.176 at infinite acceleration. In Sec. VI we discuss the physical implications of the results in the context of black-hole physics.

II. GHZ STATE

In this section we assume that Alice, Bob, and Charlie share initially the GHZ state defined as

$$|\mathrm{GHZ}\rangle_{ABC} = \frac{1}{\sqrt{2}}[|000\rangle_{ABC} + |111\rangle_{ABC}]. \tag{2.1}$$

After sharing his own qubit, Charlie moves with respect to Alice and Bob with a uniform acceleration a. Then, Charlie's vacuum and one-particle states $|0\rangle_M$ and $|1\rangle_M$, where the subscript M stands for Minkowski, are transformed into [6]

$$\begin{aligned}|0\rangle_M &\rightarrow \frac{1}{\cosh r}\sum_{n=0}^{\infty}\tanh^n r|n\rangle_I|n\rangle_{II},\\ |1\rangle_M &\rightarrow \frac{1}{\cosh^2 r}\sum_{n=0}^{\infty}\tanh^n r\sqrt{n+1}|n+1\rangle_I|n\rangle_{II},\end{aligned} \tag{2.2}$$

where r is a parameter proportional to Charlie's acceleration, and $|n\rangle_I$ and $|n\rangle_{II}$ are the mode decomposition in the two causally disconnected regions in Rindler space. Equation (2.2) implies that the physical information formed initially in region I is leaked to the inaccessible region (region II) due to accelerating motion. This loss of information causes a particle detector in region I to detect a thermally average state, which is a main scenario of the Unruh effect [7].

Therefore, Charlie's acceleration transforms the GHZ state into

$$\begin{aligned}|\mathrm{GHZ}\rangle_{ABC} \rightarrow \frac{1}{\sqrt{2}\cosh r}\sum_{n=0}^{\infty}\tanh^n r\bigg[&|00n\rangle|n\rangle_{II}\\ &+\frac{\sqrt{n+1}}{\cosh r}|11n+1\rangle|n\rangle_{II}\bigg],\end{aligned} \tag{2.3}$$

where $|abc\rangle = |ab\rangle_{AB}^M \otimes |c\rangle_I$. Since $|\psi\rangle_{II}$ is a physically inaccessible state from Alice, Bob, and Charlie, we should average it out via a partial trace. Thus, the quantum state shared by Alice, Bob, and Charlie reduces to the following mixed state:

$$\begin{aligned}\rho_{\mathrm{GHZ}} = \frac{1}{2\cosh^2 r}\sum_{n=0}^{\infty}\tanh^{2n} r\bigg[&|00n\rangle\langle 00n|\\ &+\frac{\sqrt{n+1}}{\cosh r}\{|00n\rangle\langle 11n+1| + |11n+1\rangle\langle 00n|\}\\ &+\frac{n+1}{\cosh^2 r}|11n+1\rangle\langle 11n+1|\bigg].\end{aligned} \tag{2.4}$$

This is very similar to the information loss of Hawking radiation in the black-hole physics, where the pure "in" state becomes the thermally mixed "out" state due to gravitation collapse [21].

To quantify how much ρ_{GHZ} is entangled, we introduce a π-tangle [14] defined as

$$\pi = \frac{\pi_A + \pi_B + \pi_C}{3}, \tag{2.5}$$

where

$$\begin{aligned}\pi_A &= \mathcal{N}_{A(BC)}^2 - \mathcal{N}_{AB}^2 - \mathcal{N}_{AC}^2,\\ \pi_B &= \mathcal{N}_{B(CA)}^2 - \mathcal{N}_{BC}^2 - \mathcal{N}_{BA}^2,\\ \pi_C &= \mathcal{N}_{C(AB)}^2 - \mathcal{N}_{CA}^2 - \mathcal{N}_{CB}^2.\end{aligned} \tag{2.6}$$

In Eq. (2.6), $\mathcal{N}_{\alpha(\beta\gamma)}$ and $\mathcal{N}_{\alpha\beta}$ are one-tangle and two-tangle, respectively, defined as $\mathcal{N}_{\alpha(\beta\gamma)} \equiv ||\rho_{\mathrm{GHZ}}^{T_\alpha}|| - 1$ and $\mathcal{N}_{\alpha\beta} \equiv ||(\mathrm{tr}_\gamma \rho_{\mathrm{GHZ}})^{T_\alpha}|| - 1$. Here T_α denotes the partial transposition for the α qubit, and $||A||$ is a trace norm of operator A defined as $||A|| \equiv \mathrm{tr}\sqrt{AA^\dagger}$.

Although one-tangle can be easily computed in the qubit system by using $\mathcal{N}_{A(BC)}^2 = 4\det\rho^A$, where $\rho^A = \mathrm{tr}_{BC}\,\rho_{ABC}$, we cannot use this convenient formula because Charlie's accelerating motion makes the quantum state an infinite-dimensional qudit state. Thus, we have to use the original definition for computation of one-tangle.

Now, let us compute one-tangles. In order to compute $\mathcal{N}_{A(BC)}$ first we should derive $\rho_{\mathrm{GHZ}}^{T_A}$, which is

$$\begin{aligned}\rho_{\mathrm{GHZ}}^{T_A} = \frac{1}{2\cosh^2 r}\sum_{n=0}^{\infty}\tanh^{2n} r\bigg[&|00n\rangle\langle 00n|\\ &+\frac{\sqrt{n+1}}{\cosh r}\{|10n\rangle\langle 01n+1| + |01n+1\rangle\langle 10n|\}\\ &+\frac{n+1}{\cosh^2 r}|11n+1\rangle\langle 11n+1|\bigg].\end{aligned} \tag{2.7}$$

From $\rho_{\mathrm{GHZ}}^{T_A}$ it is straightforward to derive $(\rho_{\mathrm{GHZ}}^{T_A})(\rho_{\mathrm{GHZ}}^{T_A})^\dagger$, whose matrix representation is a diagonal one. Thus, it is simple to show that the eigenvalues of $(\rho_{\mathrm{GHZ}}^{T_A})(\rho_{\mathrm{GHZ}}^{T_A})^\dagger$ are

$$\left\{\frac{\tanh^{4n} r}{4\cosh^4 r}, \frac{(n+1)\tanh^{4n} r}{4\cosh^6 r}, \frac{(n+1)\tanh^{4n} r}{4\cosh^6 r}, \frac{(n+1)^2\tanh^{4n} r}{4\cosh^8 r}\bigg| n = 0,1,2,\ldots\right\}. \tag{2.8}$$

Using Eq. (2.8) one can compute the one-tangle $\mathcal{N}_{A(BC)}$ by making use of its original definition $||\rho_{\mathrm{GHZ}}^{T_A}|| - 1$, which is

$$\mathcal{N}_{A(BC)} = \frac{1}{\cosh^3 r} \sum_{n=0}^{\infty} \sqrt{n+1} \tanh^{2n} r. \tag{2.9}$$

When deriving Eq. (2.9) we used the following formulas

$$\begin{aligned} \sum_{n=0}^{\infty} \tanh^{2n} r &= \cosh^2 r, \\ \sum_{n=0}^{\infty} (n+1) \tanh^{2n} r &= \cosh^4 r. \end{aligned} \tag{2.10}$$

Introducing a polylogarithm function $\mathrm{Li}_n(z)$ defined as

$$\mathrm{Li}_n(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n} = \frac{z}{1^n} + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots, \tag{2.11}$$

one can express $\mathcal{N}_{A(BC)}$ as

$$\mathcal{N}_{A(BC)} = \frac{1}{\sinh^2 r \cosh r} \mathrm{Li}_{-1/2}(\tanh^2 r). \tag{2.12}$$

By repeating the calculation, one can show $\mathcal{N}_{B(AC)} = \mathcal{N}_{A(BC)}$, which is, in fact, obvious by considering a symmetry of the GHZ state.

Now, let us compute the one-tangle $\mathcal{N}_{C(AB)}$. After deriving $\rho_{\mathrm{GHZ}}^{T_C}$ from ρ_{GHZ} given in Eq. (2.4), one can construct $(\rho_{\mathrm{GHZ}}^{T_C})(\rho_{\mathrm{GHZ}}^{T_C})^\dagger$, whose explicit expression is

$$\begin{aligned} \left(\rho_{\mathrm{GHZ}}^{T_C}\right)\left(\rho_{\mathrm{GHZ}}^{T_C}\right)^\dagger = \frac{1}{4\cosh^4 r} \sum_{n=0}^{\infty} \tanh^{4n} r \Bigg[\left(1 + \frac{n\cosh^2 r}{\sinh^4 r}\right) |00n\rangle\langle 00n| + \left(\frac{n+1}{\cosh^2 r} + \frac{n^2}{\sinh^4 r}\right) |11n\rangle\langle 11n| \\ + \sqrt{n+1} \left(\frac{\sinh^2 r}{\cosh^3 r} + \frac{n}{\cosh r \sinh^2 r}\right) \{ |00n+1\rangle\langle 11n| + |11n\rangle\langle 00n+1| \} \Bigg]. \end{aligned} \tag{2.13}$$

Unlike the previous cases, the matrix representation of $(\rho_{\mathrm{GHZ}}^{T_C})(\rho_{\mathrm{GHZ}}^{T_C})^\dagger$ is not a diagonal matrix. However, one can compute the eigenvalues of $(\rho_{\mathrm{GHZ}}^{T_C})(\rho_{\mathrm{GHZ}}^{T_C})^\dagger$ analytically by ordering the basis of Hilbert space as

$$\{ |000\rangle, |110\rangle, |001\rangle, |111\rangle, |002\rangle, |112\rangle, \ldots, |010\rangle, |100\rangle, |011\rangle, |101\rangle, |012\rangle, |102\rangle, \ldots \}. \tag{2.14}$$

Then, the nonvanishing eigenvalues of $(\rho_{\mathrm{GHZ}}^{T_C})(\rho_{\mathrm{GHZ}}^{T_C})^\dagger$ are

$$\left\{ \frac{1}{4\cosh^4 r}, \lambda_n^\pm \quad (n = 0,1,2,\ldots) \right\}, \tag{2.15}$$

where

$$\lambda_n^\pm = \frac{\tanh^{4n} r}{8\cosh^4 r} \left[\left\{ \frac{2(n+1)}{\cosh^2 r} + \frac{n^2}{\sinh^4 r} + \tanh^4 r \right\} \pm \sqrt{ \left\{ \frac{2(n+2)}{\cosh^2 r} + \frac{n^2}{\sinh^4 r} + \tanh^4 r \right\} \left\{ \frac{2n}{\cosh^2 r} + \frac{n^2}{\sinh^4 r} + \tanh^4 r \right\} } \right]. \tag{2.16}$$

Thus, the one-tangle $\mathcal{N}_{C(AB)}$ can be computed straightforwardly from its definition, whose explicit expression is

$$\mathcal{N}_{C(AB)} = \left|\left|\rho_{\mathrm{GHZ}}^{T_C}\right|\right| - 1 = \frac{1}{2\cosh^2 r} + \sum_{n=0}^{\infty} \left(\sqrt{\lambda_n^+} + \sqrt{\lambda_n^-}\right) - 1. \tag{2.17}$$

It seems to be impossible to express $\mathcal{N}_{C(AB)}$ in terms of the polylogarithmic function as the previous cases.

We plot the r dependence of one-tangles in Fig. 1. All one-tangles become unity at $r = 0$, which is exactly the value of one-tangles at the rest frame. As expected from the degradation of the bipartite entanglement in the noninertial frame, all one-tangles decrease with increasing acceleration of Charlie. At $r \to \infty$, $\mathcal{N}_{C(AB)}$ goes to zero. This can be understood from the fact that Alice and Bob cannot contribute to Charlie's entanglement because of Charlie's infinite acceleration with respect to Alice and Bob. From this fact, we guess that the one-tangle $\mathcal{N}_{C(AB)}$ goes to zero when Charlie falls into a black hole, while Alice and Bob are near the event horizon of the black hole. This fact also predicts that the Coffman-Kundu-Wootters (CKW) inequality [13], $\mathcal{N}_{C(AB)}^2 \geqslant \mathcal{N}_{CA}^2 + \mathcal{N}_{CB}^2$, is saturated at $r \to \infty$. As will be shown shortly, this is indeed the case. The surprising fact is that the one-tangles $\mathcal{N}_{A(BC)}$ and $\mathcal{N}_{B(CA)}$ do not vanish but go to $\sqrt{\pi}/2 \sim 0.886$ in the $r \to \infty$ limit. Mathematically, this limiting value originates from particular properties of the polylogarithmic function. Although we can understand that this limiting value is a remnant of entanglement between Alice and Bob without Charlie, we do not know why the remnant is equal to this particular value $\sqrt{\pi}/2$.

Now, let us compute two-tangles. Since $\rho_{\mathrm{GHZ}}^{AB} \equiv \mathrm{tr}_C\, \rho_{\mathrm{GHZ}} = (1/2)(|00\rangle\langle 00| + |11\rangle\langle 11|)$, it is easy to show that

$$\mathcal{N}_{AB} = \left|\left|\left(\rho_{\mathrm{GHZ}}^{AB}\right)^{T_A}\right|\right| - 1 = 0. \tag{2.18}$$

Since $\rho_{\mathrm{GHZ}}^{AC} = \rho_{\mathrm{GHZ}}^{BC}$, $\mathcal{N}_{AC}$ should be equal to $\mathcal{N}_{BC}$. One can show that the eigenvalues of $(\rho_{\mathrm{GHZ}}^{AC})^{T_A}(\rho_{\mathrm{GHZ}}^{AC})^{T_A\dagger}$ are

$$\left\{ \frac{\tanh^{4n} r}{4\cosh^4 r}, (n+1)^2 \frac{\tanh^{4n} 4}{4\cosh^8 r} \middle| n = 0,1,2,\ldots \right\}. \tag{2.19}$$

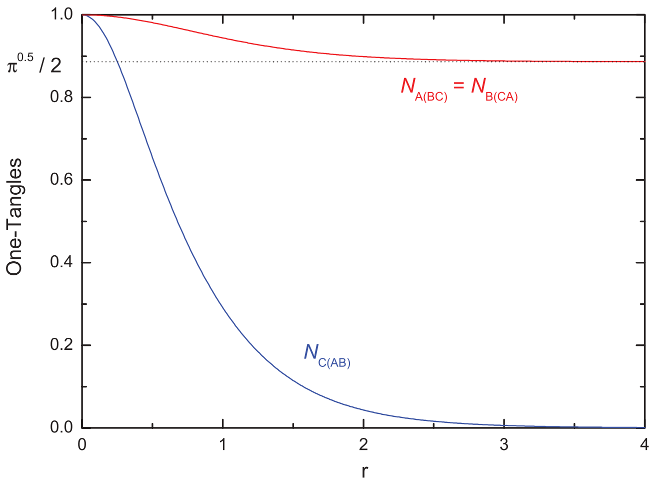

FIG. 1. (Color online) The r dependence of the one-tangles when Alice, Bob, and Charlie share the GHZ state initially. All one-tangles exhibit a decreasing behavior with increasing r. This figure shows that while $\mathcal{N}_{C(AB)}$ reduces to zero at the $r \to \infty$ limit, other one-tangles do not completely vanish but go to $\sqrt{\pi}/2 \sim 0.886$ in this limit.

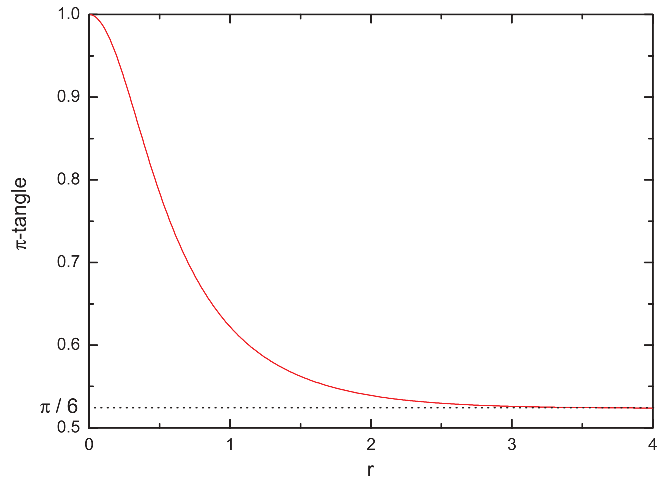

FIG. 2. (Color online) The r dependence of the π-tangle when Alice, Bob, and Charlie share the GHZ state initially. This figure indicates that the π-tangle does not vanish, but reduces to $\pi/6 \sim 0.524$ in the $r \to \infty$ limit. The physical implications of this result are discussed in the final section of this paper.

Using Eq. (2.10), therefore, one can show easily that

$$\mathcal{N}_{AC} = \mathcal{N}_{BC} = 0. \tag{2.20}$$

Thus, the two-tangles do not change in spite of Charlie's accelerating motion.

Figure 2 is a plot of r dependence of π-tangle when Alice, Bob, and Charlie share initially the GHZ state. Like bipartite entanglement, the π-tangle decreases as Charlie's acceleration increases. Unlike bipartite entanglement, however, the π-tangle does not completely vanish in the $r \to \infty$ limit but approaches $\pi/6 \sim 0.524$ in this limit. This fact enables us to predict that the tripartite entanglement does not completely vanish when Charlie falls into a black hole. If so, this is a very surprising result because this fact implies that the quantum communication process might be possible between parties even in the presence of the event horizon. This prediction should be checked in the near future by incorporating quantum-information theories into black-hole physics.

III. *W* STATE

In this section we assume that initially Alice, Bob, and Charlie share the W state

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|001\rangle_{ABC} + |010\rangle_{ABC} + |100\rangle_{ABC}). \tag{3.1}$$

By making use of Eq. (2.2) one can show that after Charlie's accelerating motion, $|W\rangle_{ABC}$ reduces to

$$|W\rangle_{ABC} \to \frac{1}{\sqrt{3}\cosh r}\sum_{n=0}^{\infty}\tanh^n r\left[\frac{\sqrt{n+1}}{\cosh r}|00n+1\rangle + |01n\rangle + |10n\rangle\right] \otimes |n\rangle_{II}, \tag{3.2}$$

where $|abc\rangle = |ab\rangle_{AB}^M \otimes |c\rangle_I$. Then, a partial trace over $|\psi\rangle_{II}$ transforms the W state into the following mixed state:

$$\rho_W = \frac{1}{3\cosh^2 r}\sum_{n=0}^{\infty}\tanh^{2n} r\left[\frac{n+1}{\cosh^2 r}|00n+1\rangle\langle 00n+1| + |01n\rangle\langle 01n| + |10n\rangle\langle 10n| + \frac{\sqrt{n+1}}{\cosh r}\{|00n+1\rangle\langle 01n| + |01n\rangle \times \langle 00n+1| + |00n+1\rangle\langle 10n| + |10n\rangle\langle 00n+1|\} + \{|01n\rangle\langle 10n| + |10n\rangle\langle 01n|\}\right]. \tag{3.3}$$

Now, let us compute two-tangles. Since $\rho_W^{AB} \equiv \mathrm{tr}_C \rho_W$ becomes

$$\rho_W^{AB} = \tfrac{1}{3}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01|), \tag{3.4}$$

it is easy to show that

$$\mathcal{N}_{AB} = \left|\left|\left(\rho_W^{AB}\right)^{T_A}\right|\right| - 1 = \frac{\sqrt{5}-1}{3}. \tag{3.5}$$

Thus, $\mathcal{N}_{AB}$ is independent of Charlie's acceleration. In order to compute $\mathcal{N}_{AC}$ we should derive ρ_W^{AC}, which can be easily derived from ρ_W by taking a partial trace over Bob's qubit. Then, it is straightforward to show that

$$\left(\rho_W^{AC}\right)^{T_A}\left(\rho_W^{AC}\right)^{T_A\dagger} = \sum_{n=0}^{\infty}[a_n|0n\rangle\langle 0n| + b_n|1n\rangle\langle 1n| + c_n\{|0n\rangle \times \langle 1n+1| + |1n+1\rangle\langle 0n|\}], \tag{3.6}$$

where

$$a_n = \frac{\tanh^{4n} r}{9\cosh^4 r}\left(1 + \frac{n+1}{\cosh^2 r} + \frac{2n}{\sinh^2 r} + \frac{n^2}{\sinh^4 r}\right),$$
$$b_n = \frac{\tanh^{4n} r}{9\cosh^4 r}\left(1 + \frac{n\cosh^2 r}{\sinh^4 r}\right), \tag{3.7}$$
$$c_n = \frac{\sqrt{n+1}\tanh^{4n} r}{9\cosh^5 r}\left(1 + \tanh^2 r + \frac{n}{\sinh^2 r}\right).$$

Although the matrix representation of $(\rho_W^{AC})^{T_A}(\rho_W^{AC})^{T_A\dagger}$ is not a diagonal one, we can compute the eigenvalues of it analytically by choosing the order of basis appropriately. Then, the nonvanishing eigenvalues of $(\rho_W^{AC})^{T_A}(\rho_W^{AC})^{T_A\dagger}$ are

$$\{b_0, \tilde{\lambda}^{\pm}|n = 0,1,2,\ldots\}, \tag{3.8}$$

where

$$\tilde{\lambda}^{\pm} = \tfrac{1}{2}\left[(a_n + b_{n+1}) \pm \sqrt{(a_n - b_{n+1})^2 + 4c_n^2}\right]. \tag{3.9}$$

Therefore, $\mathcal{N}_{AC}$ becomes

$$\mathcal{N}_{AC} \equiv \left|\left|\left(\rho_W^{AC}\right)^{T_A}\right|\right| - 1 = \sqrt{b_0} + \sum_{n=0}^{\infty}\left(\sqrt{\tilde{\lambda}_n^+} + \sqrt{\tilde{\lambda}_n^-}\right) - 1. \tag{3.10}$$

Since $\rho_W^{BC} \equiv \mathrm{tr}_A \rho_W$ is equal to ρ_W^{AC}, it is easy to show that $\mathcal{N}_{BC} = \mathcal{N}_{AC}$.

The r dependence of the two-tangles is plotted in Fig. 3. When Charlie's acceleration is zero, all two-tangles become $(\sqrt{5}-1)/3 \sim 0.412$, which is two-tangle in the rest frame. As shown in Eq. (3.5) $\mathcal{N}_{AB}$ is independent of r. This is because the contribution of Charlie's qubit averages out via the partial trace in ρ_W^{AB}. However, $\mathcal{N}_{AC}$ and $\mathcal{N}_{BC}$ decrease with increasing r. This implies that the effect of Charlie's acceleration is contributed to these two two-tangles. The remarkable fact is that $\mathcal{N}_{AC}$ and $\mathcal{N}_{BC}$ become almost zero at $r \geqslant 0.89$. This brings back a concurrence [22], entanglement measure for bipartite quantum state, which is defined as $\max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$, where λ_i are the eigenvalues, in decreasing order, of the Hermitian operator $\sqrt{\sqrt{\rho}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\sqrt{\rho}}$.

Now, let us compute one-tangles. In order to compute $\mathcal{N}_{A(BC)}$ we should derive $\rho_W^{T_A}$, which can be read directly from ρ_W by taking the partial transposition for Alice's qubit. Then, after some algebra, it is straightforward to show that

$$\begin{aligned}\left(\rho_W^{T_A}\right)\left(\rho_W^{T_A}\right)^{\dagger} = \lim_{N\to\infty}\sum_{n=0}^{N}\big[&\bar{a}_n|00n\rangle\langle 00n| + \bar{b}_n|01n\rangle\langle 01n| + \bar{c}_n|10n\rangle\langle 10n| + \bar{d}_n|11n\rangle\langle 11n| + \bar{f}_n\{|00n+1\rangle\langle 01n| + |01n\rangle\langle 00n+1|\}\\ &+ \bar{g}_n\{|00n\rangle\langle 10n+1| + |10n+1\rangle\langle 00n|\} + \bar{h}_n\{|00n\rangle\langle 11n| + |11n\rangle\langle 00n|\} + \bar{j}_n\{|01n\rangle\langle 10n+2| + |10n+2\rangle\langle 01n|\}\\ &+ \bar{k}_n\{|01n\rangle\langle 11n+1| + |11n+1\rangle\langle 01n|\} + \bar{\ell}_n\{|10n+1\rangle\langle 11n| + |11n\rangle\langle 10n+1|\}\big],\end{aligned} \tag{3.11}$$

where

$$\bar{a}_n = \frac{\tanh^{4n} r}{9\cosh^4 r}\left(1 + \frac{n+1}{\cosh^2 r} + \frac{n^2 + n\cosh^2 r}{\sinh^4 r}\right),$$
$$\bar{b}_n = \frac{\tanh^{4n} r}{9\cosh^4 r}\left(1 + \frac{n+1}{\cosh^2 r}\right),$$
$$\bar{c}_n = \frac{\tanh^{4n} r}{9\cosh^4 r}\left(1 + \frac{n\cosh^2 r}{\sinh^4 r}\right),$$
$$\bar{d}_n = \frac{\tanh^{4n} r}{9\cosh^4 r},$$
$$\bar{f}_n = \frac{\sqrt{n+1}\tanh^{4n} r}{9\cosh^5 r}\left(1 + \frac{n+1}{\cosh^2 r}\right), \tag{3.12}$$
$$\bar{g}_n = \frac{\sqrt{n+1}\tanh^{4n} r}{9\cosh^5 r}\left(\tanh^2 r + \frac{n}{\sinh^2 r}\right),$$
$$\bar{h}_n = \frac{n\tanh^{4n} r}{9\cosh^4 r\sinh^2 r},$$
$$\bar{j}_n = \frac{\sqrt{(n+1)(n+2)}\tanh^{4n} r}{9\cosh^8 r}\sinh^2 r,$$
$$\bar{k}_n = \frac{\sqrt{n+1}\tanh^{4n} r}{9\cosh^7 r}\sinh^2 r,$$
$$\bar{\ell}_n = \frac{\sqrt{n+1}\tanh^{4n} r}{9\cosh^5 r}.$$

It does not seem to be possible to compute the eigenvalues of $(\rho_W^{T_A})(\rho_W^{T_A})^{\dagger}$ analytically. Thus, we adopt the following numerical procedure for the calculation of the eigenvalues. First, we take $N = 256$ in Eq. (3.11) and compute numerically $\eta(N,r) = \sum_{i=1}^{N}\sqrt{\lambda_i} - 1$, where λ_i are the eigenvalues of

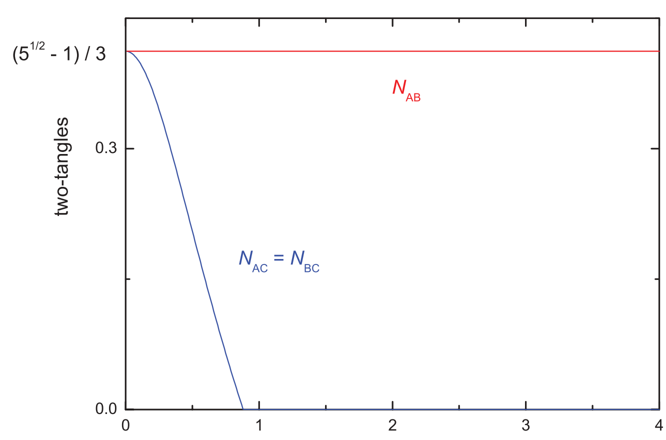

FIG. 3. (Color online) The r-dependence of the two-tangles when Alice, Bob, and Charlie share the W state initially. This figure shows that the two-tangle $\mathcal{N}_{AB}$ is independent of Charlie's acceleration. However $\mathcal{N}_{AC}$ and $\mathcal{N}_{BC}$ decrease with increasing r and become zero at $r \geqslant 0.89$.

$(\rho_W^{T_A})(\rho_W^{T_A})^\dagger$ and $N = 256$. The large N behavior of $\eta(N,r)$ can be computed by a numerical fitting method using $\eta(256,r)$. Since $\mathcal{N}_{A(BC)} = \lim_{N\to\infty}\eta(N,r)$, the r dependence of $\mathcal{N}_{A(BC)}$ can be computed by following this procedure. The result of the numerical calculation is shown in Fig. 4. As Fig. 4 exhibits, $\mathcal{N}_{A(BC)}$ becomes $2\sqrt{2}/3 \sim 0.943$ at $r = 0$. This is a value of one-tangle for the W state in the rest frame. As expected, it monotonically decreases with increasing r, but does not completely vanish at $r \to \infty$ limit. In this limit $\mathcal{N}_{A(BC)}$ reduces to $\mathcal{N}_{A(BC)} \sim 0.659$, which is smaller than $\sqrt{\pi}/2 \sim 0.886$, the corresponding value for the GHZ state.

In order to compute $\mathcal{N}_{B(AC)}$ we should derive $(\rho_W^{T_B})(\rho_W^{T_B})^\dagger$, which can be derived straightforwardly from $\rho_W^{T_B}$. The final expression of $(\rho_W^{T_B})(\rho_W^{T_B})^\dagger$ is

$$\begin{aligned}(\rho_W^{T_B})(\rho_W^{T_B})^\dagger = \lim_{N\to\infty}\sum_{n=0}^{N}[&\bar{a}_n|00n\rangle\langle 00n| + \bar{c}_n|01n\rangle\langle 01n| + \bar{b}_n|10n\rangle\langle 10n| + \bar{d}_n|11n\rangle\langle 11n| + \bar{f}_n\{|00n+1\rangle\langle 10n| + |10n\rangle\langle 00n+1|\} \\ &+ \bar{g}_n\{|00n\rangle\langle 01n+1| + |01n+1\rangle\langle 00n|\} + \bar{h}_n\{|00n\rangle\langle 11n| + |11n\rangle\langle 00n|\} + \bar{\ell}_n\{|01n+1\rangle\langle 11n| + |11n\rangle\langle 01n+1|\} \\ &+ \bar{j}_n\{|01n+2\rangle\langle 10n| + |10n\rangle\langle 01n+2|\} + \bar{k}_n\{|10n\rangle\langle 11n+1| + |11n+1\rangle\langle 10n|\}],\end{aligned} \tag{3.13}$$

where the coefficients are given in Eq. (3.12). Since $(\rho_W^{T_B})(\rho_W^{T_B})^\dagger$ can be obtained from $(\rho_W^{T_A})(\rho_W^{T_A})^\dagger$ by interchanging Alice's qubit and Bob's qubit, the eigenvalues of $(\rho_W^{T_B})(\rho_W^{T_B})^\dagger$ should be equal to those of $(\rho_W^{T_A})(\rho_W^{T_A})^\dagger$. Thus we have $\mathcal{N}_{B(CA)} = \mathcal{N}_{A(BC)}$.

Finally, we compute $\mathcal{N}_{C(AB)}$. Since $(\rho_W^{T_C})(\rho_W^{T_C})^\dagger$ becomes

$$\begin{aligned}(\rho_W^{T_C})(\rho_W^{T_C})^\dagger = \sum_{n=0}^{\infty}[&\tilde{a}_n|00n\rangle\langle 00n| + \tilde{b}_n\{|01n\rangle\langle 01n| + |10n\rangle\langle 10n| + |01n\rangle\langle 10n| + |10n\rangle\langle 01n|\} + \tilde{c}_n\{|00n\rangle\langle 01n+1| \\ &+ |01n+1\rangle\langle 00n| + |00n\rangle\langle 10n+1| + |10n+1\rangle\langle 00n|\}],\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\tilde{a}_n &= \frac{\tanh^{4n} r}{9\cosh^4 r}\left(\frac{2(n+1)}{\cosh^2 r} + \frac{n^2}{\sinh^4 r}\right), \\ \tilde{b}_n &= \frac{\tanh^{4n} r}{9\cosh^4 r}\left(2 + \frac{n\cosh^2 r}{\sinh^4 r}\right), \\ \tilde{c}_n &= \frac{\tanh^{4n} r}{9\cosh^4 r}\left(\frac{2\sqrt{n+1}\sinh^2 r}{\cosh^3 r} + \frac{n\sqrt{n+1}}{\sinh^2 r\cosh r}\right),\end{aligned} \tag{3.15}$$

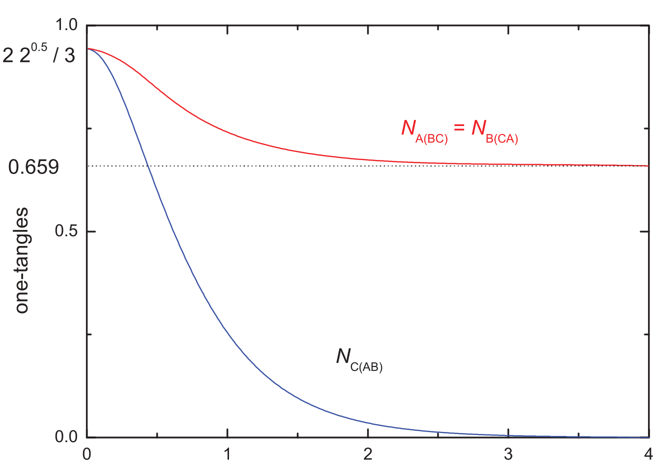

FIG. 4. (Color online) The r dependence of the one-tangles when Alice, Bob, and Charlie share the W state initially. Like the case of GHZ state, all one-tangles exhibit a decreasing behavior with increasing r. This figure shows that while $\mathcal{N}_{C(AB)}$ reduces to zero at $r \to \infty$ limit, other one-tangles do not completely vanish but go to 0.659 in this limit.

it is not difficult to compute the eigenvalues of $(\rho_W^{T_C})(\rho_W^{T_C})^\dagger$ analytically by choosing the order of the basis appropriately. The final expression of the eigenvalues is

$$\{2\tilde{b}_0, \Lambda_n^{\pm}|n = 0,1,2,\ldots\}, \tag{3.16}$$

where

$$\Lambda_n^{\pm} = \tfrac{1}{2}\left[(\tilde{a}_n + 2\tilde{b}_{n+1}) \pm \sqrt{(\tilde{a}_n - 2\tilde{b}_{n+1})^2 + 8\tilde{c}_n^2}\right]. \tag{3.17}$$

Therefore, $\mathcal{N}_{C(AB)}$ is given by

$$\mathcal{N}_{C(AB)} = \sqrt{2\tilde{b}_0} + \sum_{n=0}^{\infty}\left(\sqrt{\Lambda_n^+} + \sqrt{\Lambda_n^-}\right) - 1. \tag{3.18}$$

The r dependence of the one-tangles are plotted at Fig. 4. Like the GHZ case, all one-tangles decrease with increasing r. While $\mathcal{N}_{C(AB)}$ goes to zero in $r \to \infty$ limit, $\mathcal{N}_{A(BC)}$ and $\mathcal{N}_{B(CA)}$ do not completely vanish but reduce to 0.659 in this limit. This value is smaller than the corresponding remnant $\sqrt{\pi}/2$ of the one-tangles for the GHZ state. As we commented in the previous section, we do not know why the remnant of $\mathcal{N}_{A(BC)} = \mathcal{N}_{B(CA)}$ is 0.659.

The r dependence of π-tangle for W state is plotted in Fig. 5. As expected, π_A, π_B, and π_C decrease with increasing r from $4(\sqrt{5}-1)/9 \sim 0.55$, which is a corresponding value at $r = 0$. While π_C goes to zero at $r \to \infty$ limit, π_A and π_B do not completely vanish in this limit, but reduce to 0.265. For this reason π_W, the π-tangle of the W state, becomes 0.176 when Charlie moves with respect to Alice and Bob with infinite acceleration. The remnant 0.176 for the W state is much smaller than the corresponding value $\pi/6 \sim 0.524$ for the GHZ state. We do not clearly understand why the remnant of π-tangle for the W state is much smaller than that for the GHZ state. We also do not understand why the tripartite

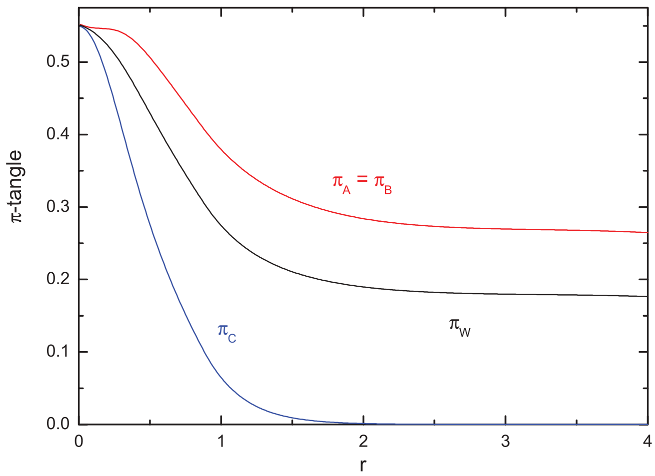

FIG. 5. (Color online) The r dependence of the π-tangle when Alice, Bob, and Charlie share the W state initially. Like the GHZ state, the π-tangle π_W exhibits a monotonically decreasing behavior with increasing r and reduces to 0.176 at $r \to \infty$ limit. Mathematically, this is due to the fact that π_A and π_B become nonzero while π_C reduces to zero in this limit. The physical implications of this result are discussed in Sec. IV.

entanglement is not zero even when Charlie approaches the Rindler horizon.

IV. CONCLUSION

In this paper we consider tripartite entanglement when one of the parties moves with uniform acceleration with respect to other parties. The accelerating motion of the one party is described by the Rindler coordinate. We adopt the π-tangle as a measure of the tripartite entanglement solely due to its calculational easiness.

Since the Unruh effect predicts that the information formed in some region in Rindler space is leaked into the causally disconnected region due to acceleration of one party, we expect that the tripartite entanglement decreases with increasing acceleration, and eventually reduces to zero at the infinite acceleration limit like the bipartite entanglement [6]. Actual calculation reveals the monotonically decreasing behavior of the π-tangle with increasing acceleration. However, actual calculation also shows that our expectation that the π-tangle vanishes in the infinite acceleration limit is wrong. If, for example, the three parties share the GHZ state initially, the corresponding π-tangle decreases monotonically from 1 at zero acceleration to $\pi/6 \sim 0.524$ at infinite acceleration. If the parties share the W state initially, the π-tangle also decreases monotonically from $4(\sqrt{5}-1)/9 \sim 0.55$ at zero acceleration to 0.176 at infinite acceleration. Thus, the π-tangle does not completely vanish even if one of the parties approaches the Rindler horizon.

The nonvanishing of the π-tangle at infinite acceleration is a striking result. Since Rindler space-time is similar to the Schwarzschild space-time, this result enables us to conjecture that the tripartite entanglement does not completely vanish even if one party falls into the event horizon of the black hole. If so, some quantum-information processing such as tripartite teleportation [23] can be performed between the inside and outside of the black hole. Since, however, the Rindler horizon is physically different from the event horizon, we should check this conjecture explicitly by actual calculation. We would like to revisit this issue in the near future.

Probably, the nonvanishing of the π-tangle at infinite acceleration is due to the incomplete definition of the π-tangle as a measure of the tripartite entanglement. Thus, it seems to be interesting to repeat the calculation of this paper by making use of the three-tangle. Since, however, the computation of the three-tangle requires the optimal decomposition of the given mixed state, its calculation is much more difficult than that of the π-tangle. In order to explore this issue, therefore, we should develop analytical and numerical techniques for the computation of the three-tangle.

Since recent string and brane-world theories predict the extra dimensions in space-time, it seems to be also of interest to study the effect of the extra dimensions in the degradation phenomena of bipartite and tripartite entanglements. Another interesting issue is to explore the effect of the black hole's rotation in the bipartite and tripartite entanglements. There are many interesting questions related to this issue. For example, it would be interesting to examine the relation between super-radiance and degradation of entanglement. We hope to explore these issues in the future.

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