Entanglement for Multipartite Quantum States

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Abstract

Entanglement for Multipartite Quantum States

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In this thesis we have reviewed the recent results on the pure- and mixed-state entanglement. We have shown explicitly that the geometric and Groverian entanglement measures for the three-qubit pure states can be expressed in terms of some geometric quantities of polygons constructed by the parameters of given quantum state. The geometric interpretation of the quantum entanglement may shed light on the profound meaning on the multipartite entanglement. We have discussed this issue in detail. For mixed-state entanglement we have developed the calculational technique for the residual entanglement or three-tangle by considering the rank-3 mixture composed of Greenberger-Horne-Zeilinger (GHZ)-, W- and flipped W-states. We also have shown that in addition to W state the three-tangle does not properly quantify the tripartite entanglement of mixed state composed of only GHZ-states. This means that some mixtures composed of only GHZ-states can be expressed in terms of only W-states. This surprising result may shed light on the fact that the set of the mixed W-states is not of measure zero in the set of whole three-qubit mixed states.

Chapter 1

Introduction

Research into entanglement of quantum states has a long history, and approximately initiated from the beginning of quantum mechanics[1, 2]. Especially, the famous paper written by Einstein, Podolsky, and Rosen (EPR) discussed carefully the validity of the quantum mechanics as a theory which can explain the physical reality. They asserted that entanglement of the quantum states makes the quantum mechanics to be non-local theory and therefore, it is not a complete theory. The EPR's claim is based on their view of nature that the theory which can explain the laws of nature should be local. Nowadays, their argument is known as EPR paradox and still the validity of the paradox is intensively discussed by many theoretical and experimental scientists.

After 30 years from the EPR's original paper J. S. Bell re-examined the EPR argument. Firstly, he accepted the EPR's argument that quantum description of the physical reality is not complete. In order to make the quantum theory local, therefore, he developed a concept of the local hidden variable (LHV). Bell proved that the introduction of LHV imposes a constraint on statistical correlation. Nowadays this constraint is called Bell's inequality[3, 4]. Therefore, any local theory should obey the Bell's inequality, but suitable measurements for some entangled quantum states does not obey the Bell's inequality. Therefore, the agreement or disagreement of the Bell's inequality becomes a cornerstone of the local or non-local theories.

Later, violation of the Bell's inequalities has been demonstrated using the polarization of photons[5, 6]. It turned out that experimental results are excellently in agreement with the predictions of quantum mechanics. Apart from photon's polarization the violation of Bell's inequalities has been demonstrated by using photon's entanglement based on position and time[7, 8], phase and momentum[9], and orbital angular momentum[10]. Also, there were several more experiments, which have not used photon but used proton[11] and atoms[12]. It is worthwhile noting that to date no experiment has been loophole free, so that technically, EPR argument on local realism has not been ruled out experimentally[13, 14].

Although entanglement has been studied for a long time, a flurry of much activities on the entanglement was initiated from early 80's in the context of quantum information theories[15]. In 1982 and 1986 R. Feynman suggested a computer which performs computation with obeying the quantum mechanical rules[16, 17]. This computer is now called quantum computer. At the same time Bennett and Brassard[18] suggested the quantum cryptographic scheme, which is much more secure than the classical cryptography against the eavesdropping. This is nowadays called BB84 protocol.

The quantum computation and quantum cryptography are two major applications of the quantum information processing. Entanglement of the quantum states plays an important role in the quantum information processing. In early 90's it was shown[19] that entanglement of the quantum states[1] makes it possible to teleport an unknown quantum state to the remote receiver. It also makes it possible to send two classical bits by sending only one quantum bit, *say* qubit[20]. These two phenomena are nowadays called quantum teleportation and superdense coding respectively and they are basic tools for the quantum information processing. In 1991 Ekert[21] suggested a new quantum cryptographic protocol, which fully uses the entanglement of the two qubit states unlike BB84 protocol. In 1994 Shor[22] developed a factoring quantum algorithm by making use of the entanglement, which factors a huge number within a polynomial time. In addition, in 1996 Grover[23, 24] developed the quantum search algorithm, which enables us to find a card from arbitrary N cards within $\sqrt{N}$ steps. Recently, Shor's factoring algorithm is experimentally realized by making use of NMR[25] and optical[26, 27] set-up. The physical implementation of Grover's search algorithm also has been made by making use of NMR[28, 29], optical[30] and cavity QED[31] set-up.

From the theoretical ground Vidal[32] has shown that entanglement of the given quantum states is a genuine physical resource, which is responsible for the speed-up of the quantum computer. To understand more deeply the general properties of the entanglement one may need to quantify the entanglement[33]. Quantification of entanglement can make transition from merely quantum mechanical "notion" to

[1] this is usually called quantum channels on the contrary to classical channels such as telephone line or broadcasting.

physical real quantity. Such quantities are called entanglement measure.

The most important property which the entanglement measure should have is the monotonicity under the local operation and classical communication (LOCC)[34]. Following the axioms given in Ref.[33] many entanglement measures were constructed such as relative entropy[35], entanglement of distillation[36] and formation[37, 38, 39, 40], geometric measure[41, 42, 43, 44], Schmidt measure[45] and Groverian measure[46]. Entanglement measures are used in various branches of quantum mechanics. Especially, recently, they are used to try to understand Zamolodchikov's c-theorem[47] more profoundly. It may be an important application of the quantum information techniques to understand the effect of renormalization group in field theories[48].

Recent research into entanglement can be summarized as following.

1.0.1 Separability Criterion

It is important to determine whether a given quantum state is separable or entangled. Until now several criteria are developed such as positive partial transpose (PPT) criterion[49, 50], reduction criterion[51], and majorization criterion[52].

The PPT criterion can be summarized as follows:

if the given quantum state ρ is separable, its partial transpose, say ρ^{T_A} is positive operator.

It was shown in Ref.[50] that for bipartite systems the converse (*i.e. if $\rho^{T_A} \geq 0$, then ρ is separable*) is true only for low-dimensional systems, namely for composite states of dimension 2×2 or 2×3. For higher dimensions it is only necessary condition because the existence of entangled states has been shown[53]. Such states are called bound entangled states[54].

The reduction criterion can be summarized as follows:

if the given quantum states ρ is separable, then $\rho_A \otimes \mathbb{1} - \rho \geq 0$ and $\rho_B \otimes \mathbb{1} - \rho \geq 0$, where $\rho_A = tr_B \rho$ and $\rho_B = tr_A \rho$.

Like the PPT criterion the reduction criterion is a necessary and sufficient condition only for 2×2 or 2×3, and necessary condition otherwise.

The majorization criterion can be summarized as follows:

if a given quantum state ρ is separable, then $\lambda_\rho^\downarrow \prec \lambda_{\rho_A}^\downarrow$ and $\lambda_\rho^\downarrow \prec \lambda_{\rho_B}^\downarrow$.

Here, $\lambda_\rho^\downarrow$ denotes the vector consisting of the eigenvalues of ρ, in decreasing order and a vector $x^\downarrow$ is majorized by a vector $y^\downarrow$, denoted as $x^\downarrow \prec y^\downarrow$, when $\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow$ holds for $k = 1, 2, \cdots, d-1$ and the equality holds for $k = d$, with d being the dimension of vector. The majorization criterion is only necessary, not a sufficient condition for separability.

Besides these criteria there are non-operational separability criteria such as positive map[50] and entanglement witnesses[55].

1.0.2 Entanglement Distillation

As mentioned before entanglement is a genuine physical resource for the quantum information processing. For example, quantum teleportation with noisy quantum channel makes the fidelity[56] decrease, which reduces the teleportation processing imperfect[57]. Since the effect of the environment, which prevents the quantum system from being isolated, is inevitable, one has to find a method which overcomes the noises, at least in principle. This method is called entanglement distillation.

So far there are two kinds of the distillation protocol. First one is a recurrence (or IBM) protocol developed in Ref.[37, 58]. Second method is a Quantum Privacy Amplification (or Oxford) protocol developed in Ref.[59]. Even if the methods are different in the detailed techniques, the overall method is identical: *given several copies of the non-maximally entangled states, retrieve the small number of the maximally entangled states via LOCC and appropriate measurements.* The distillation protocols are nicely reviewed in Ref.[60].

1.0.3 Classification of quantum states

There is an important question in the context of the distillation process. *Given an entangled quantum state, can its entanglement be distilled?* In general, this question is still unsolved. However, a necessary and sufficient criterion for distillability was proved in Ref.[61]. The condition can be summarized as follows:

Te state ρ is distillable iff there exists $|\psi^2\rangle = a_1|e_1\rangle|f_1\rangle + a_2|e_2\rangle|f_2\rangle$ such that $\langle\psi^2|(\rho^{T_A})^{\otimes n}|\psi^2\rangle < 0$ for some n.

In other words, if for a certain n copies the partial transpose of the total state has a negative eigenvalue with some vector of Schmidt rank 2, then ρ can be distillable (one says: ρ is n-distillable), and vice versa.

From this condition it follows immediately that a state with a PPT-entangled states cannot be distilled: *if $\rho^{T_A} \geq 0$, then $(\rho^{T_A})^{\otimes n} \geq 0$, and thus PPT-entangled states are undistillable.* This is why the entanglement of the PPT-entangled states is called “bound entanglement”.

Besides the PPT-entangled states are there any other undistillable entangled quantum states? This question is also unsolved yet. If such states exist, we call them non-positive partial transpose (NPT)-entangled states. For higher dimensions a strong conjecture that NPT-entangled states exist[62, 63]. Then, the set of the

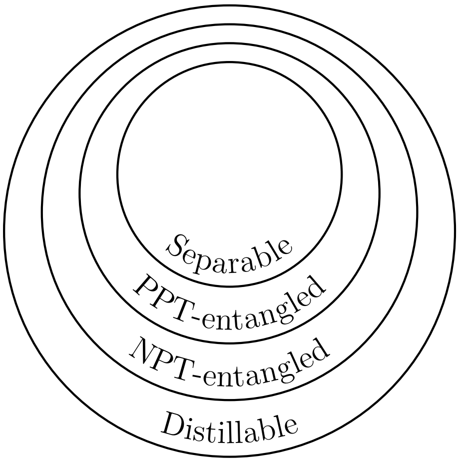

Figure 1.1: Characterization of the quantum states in terms of the distillability.

quantum states can be classified in terms of the distillability or undistillability as Fig. 1.

In this thesis we will present the recent results on the analytic derivation of the entanglement measures. In section II we will present three theorems, which plays crucial role in the following calculations performed in the subsequent sections. This section is based on Ref.[64]. In section III by making use of the theorems of the previous section we compute the geometric (and/or Groverian) entanglement measure for the 3-qubit generalized W-state[65] defined as $|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle$. It is shown in this section that the geometric entangled measure of the W-state is expressed by two different ways depending on the region of the parameter space. First expression is expressed in terms of the largest coefficient $\alpha^2 = \max(a^2, b^2, c^2)$, and second one is expressed in terms of the circumradius of the triangle (a, b, c). This section is based on Ref.[66]. In section IV we generalize section III by considering the 3-qubit state $|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle + d|111\rangle$. It is shown that the geometric measure for this state has two different expressions. As generalized W-state first expression is expressed in terms of the largest coefficient $\alpha^2 = \max(a^2, b^2, c^2, d^2)$. Second one is expressed in terms of the circumradius of the convexed quadrangle (a, b, c, d). This section is based on Ref.[67]. In section V we use the classification of the 3-qubit states suggested in Ref.[68]. Using the classification we compute the geometric measures for states in Type I, II, and III, and re-express the results in terms of the local unitary (LU)-invariants. This section is based on Ref.[69]. In section VI we try to extend the previous section by computing the geometric measure for the higher qubit systems. In this section we compute the geometric measure for the one-parametric n-qubit states

$$|W_n\rangle = a|10\cdots0\rangle + a|01\cdots0\rangle + \cdots + a|0\cdots10\rangle + q|0\cdots01\rangle$$

and two-parametric 4-qubit W-state

$$|W_4\rangle = a|1000\rangle + b|0100\rangle + q|0010\rangle + q|0001\rangle.$$

This section is based on Ref.[70].

From section VII we present the recent results on the entanglement of the mixed states. After developing the calculational techniques for the concurrence of the arbitrary two-qubit mixed states by Wootters[38, 39], this technique is used to construct the tripartite entanglement measure, so-called residual entanglement or three-tangle in Ref.[71]. In fact, the three-tangle derived in Ref.[71] exactly coincides with the modulus of a Cayley's hyperdeterminant[72, 73] and is an invariant quantity under the local $SL(2,\mathbb{C})$ transformation[74, 75]. After six years from the construction of the three-tangle in Ref.[71] Lohmayer *et al*[76] derived analytically the three-tangle for the mixture of Greenberger-Horne-Zeilinger(GHZ)[77] and W[65] states defined

$$\rho = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W| \tag{1.1}$$

where

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \qquad |W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right). \tag{1.2}$$

It was shown that in Ref.[76] that the three-tangle for the quantum state ρ given in Eq.(1.1) has three different expressions depending on the parameter p. It was also shown that the three-tangle and concurrences for the sub-systems satisfy the monogamy inequality

$$\tau_3 + \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \leq \mathcal{C}_{A(BC)}^2 \tag{1.3}$$

where τ_3 is a three-tangle and, $\mathcal{C}_{AB}^2$, $\mathcal{C}_{AC}^2$ and $\mathcal{C}_{A(BC)}^2$ are concurrences for the corresponding sub-systems. The authors in Ref.[78] extended the results of Ref.[76] by computing the three-tangle for the mixture of the generalized GHZ and generalized W states as following:

$$\rho(p) = p|gGHZ\rangle\langle gGHZ| + (1-p)|gW\rangle\langle gW| \tag{1.4}$$

where

$$|gGHZ\rangle = a|000\rangle + b|111\rangle \qquad |gW\rangle = c|001\rangle + d|010\rangle + f|100\rangle. \tag{1.5}$$

In Ref.[79] the general properties of the three-tangle are more deeply discussed by introducing two key concepts, *zero-polytope* and *characteristic curve*. Therefore, the result of Ref.[79] can be used to check whether or not the calculational result for the three-tangle is correct.

In section VII we compute the three-tangle for the rank-3 mixture, *i.e.* mixture of GHZ, W, and flipped-W states. In this section we also provide a technique, which determines whether or not an arbitrary rank-3 state has vanishing three-tangle by making use of the Bloch sphere S^8 of the qutrit system. This section is based on Ref.[80]. In section VIII we show that the three tangle does not properly quantify the tripartite entanglement of some particular mixture composed of only GHZ states. This is a surprising result, which has not been expected before. This fact may shed light on the physical reason why the set of the mixed W-states is not of measure zero in the set of whole 3-qubit mixed states, which was discussed in Ref.[81]. This section is based on Ref.[82]. In section IX we give a brief conclusion and direction of future research.

Part I

Entanglement for pure states

Chapter 2

Reduced state uniquely defines the Groverian measure of the original pure state

Quantum theory opens up new possibilities for information processing and communication and the entanglement of a quantum state allows to carry out tasks, which could not be possible with a classical system [83, 37, 36, 33, 84, 39, 85, 86]. It plays a pivotal role for exponential speedup of quantum algorithms [32], teleportation [19] and superdense coding [20].

The quantum correlation is the essence of the entanglement and it cannot be created by local operations and classical communication (LOCC) alone. Analysis of multi-particle entanglement provides insight into the nature of quantum correlation. However, current situation is far from satisfaction.

Linden *et al.* revealed that almost every pure state of three qubits is completely determined by its two-particle reduced density matrices [87]. In other words, we cannot get much new information from the given pure three-qubit state if the reduced two-qubit states are known. The case of pure states of any number n of parties was considered in Ref.[88] and it was shown that the reduced states of a fraction of the parties uniquely specify the quantum state. One may consider more general and open questions of vital importance: how much information is contained in any reduced $(n-1)$-qubit state? How do we use this information to convert the *nonlinear eigenproblem* of entanglement measure calculation to the *linear eigenproblem*? Is there any physically relevant connection between the pure n-party states which have LU-equivalent $(n-1)$-party reduced states? Does such a connection impose an upper bound for entanglement measure?

Groverian entanglement measure G [46] gives concise answers to all these questions. It is an entanglement measure defined in operational terms, namely, how well a given state serves as the input to Grover's search algorithm [24]. Groverian measure depends on maximal success probability $P_{\max}$ and is defined by the formula $G(\psi) = \sqrt{1 - P_{\max}}$. The maximal success probability is the overlap of a given state with the nearest separable state. The same overlap defines Geometric measure of entanglement introduced earlier as an axiomatic measure[41, 42, 43]. In this view Groverian measure gives an operational treatment of the axiomatic measure and is a good tool to investigate the above-mentioned questions. In the following we will consider only the maximal success probability and our conclusions are valid for both Groverian and Geometric measures.

Surprisingly enough, any reduced state resulting from a partial trace over a single qubit suffices to find $P_{\max}$ of the original pure state. For example, the entanglement of three-qubit pure state is completely understood from the two-qubit mixed state reduced from the original pure state. Since bipartite systems, regardless mixed or pure, always give a linear eigenproblem, this fact enables us to obtain analytic expressions of Groverian entanglement measures for pure three qubit states.

It is well-known that entanglement measures are invariant under local unitary transformations [33, 89, 90, 91]. However, LU-equivalent condition is not the only one for the same Groverian entanglement measure. In fact, if two pure states have LU-equivalent reduced states which are obtained by taking partial trace once, it turns out that they have same entanglement measures. Owing to this the lower bound for $P_{\max}$ is derived. However, it is not reachable for three and higher qubit states and, therefore, is not precise.

In Section 2.1 we derive a formula connecting Groverian measure of a pure state and its reduced density matrix. In Section 2.2 we establish a lower bound for Groverian measure. In Section 2.3 we present analytic expressions for the maximal success probability that reflect main features of both measures. In Section 2.4 we make concluding remarks of this chapter.

2.1 Groverian measure in terms of reduced densities

We consider a pure n-qudit state $|\psi\rangle$. The maximum probability of success is defined by

$$P_{\max}(\psi) = \max_{q_1 q_2 \cdots q_n} |\langle q_1 q_2 \dots q_n|\psi\rangle|^2, \tag{2.1}$$

where $|q_k\rangle$'s are pure single qudit normalized states. Our intention is to derive a formula which connects the maximum probability of success and $(n-1)$-qudit reduced states. In general, reduced states are mixed states and are described by density matrices. Hence we express the maximum probability of success in terms of density operators right away. We will use the notation ρ for the state $|\psi\rangle$ and ϱ for the pure single qudit state density operators, respectively. Eq.(2.1) takes the form

$$P_{\max}(\rho) = \max_{\varrho_1 \varrho_2 \cdots \varrho_n} \operatorname{tr}\left(\rho\, \varrho_1 \otimes \varrho_2 \otimes \cdots \otimes \varrho_n\right). \tag{2.2}$$

Theorem 1. *Any $(n-1)$-qudit reduced state uniquely determines the Groverian and Geometric measures of the original n-qudit pure state.*

Proof. Define a single qudit state $|\chi\rangle$ by the formula

$$|\chi\rangle = \langle q_1 q_2 \dots \widehat{q_k} \dots q_n|\psi\rangle, \tag{2.3}$$

where $\widehat{\ }$ means exclusion. Obviously

$$|\langle q_1 q_2 \dots q_n|\psi\rangle|^2 = |\langle q_k|\chi\rangle|^2 = \operatorname{tr}(|\chi\rangle\langle\chi|\varrho_k). \tag{2.4}$$

The absolute value of the inner product $|\langle q_k|\chi\rangle|$ is maximum when $q_k = |\chi\rangle/\sqrt{\langle\chi|\chi\rangle}$ and therefore

$$\max_{\varrho_k} \operatorname{tr}(|\chi\rangle\langle\chi|\varrho_k) = \langle\chi|\chi\rangle = \operatorname{tr}\left(|\chi\rangle\langle\chi|\right). \tag{2.5}$$

Denote by $\rho(\widehat{k})$ the reduced state resulting from a partial trace over k-th qudit, that is $\rho(\widehat{k}) = \operatorname{tr}_k \rho(\psi)$. From this definition it follows the identity

$$\operatorname{tr}(|\chi\rangle\langle\chi|) = \operatorname{tr}\left(\rho(\widehat{k})\varrho_1 \otimes \varrho_2 \otimes \dots \widehat{\varrho_k} \dots \otimes \varrho_n\right). \tag{2.6}$$

Owing to this identity Eq.(2.5) can be rewritten as

$$\max_{\varrho^k} \operatorname{tr}\left(\rho\, \varrho^1 \otimes \varrho^2 \otimes \dots \otimes \varrho^n\right) = \operatorname{tr}\left(\rho(\widehat{k})\varrho_1 \otimes \varrho_2 \otimes \dots \widehat{\varrho_k} \dots \otimes \varrho_n\right). \tag{2.7}$$

Both sides of the Eq.(2.7) must have the same maximum and this is the proof of the theorem.

Since the r.h.s. of Eq.(2.7) contains the reduced density operator $\mathrm{tr}_k\,\rho = \rho(\widehat{k})$ which is generally mixed state, the next maximization is nontrivial.

Eq.(2.7) does not mean that a pure state and its once reduced state have equal Groverian measures. One can not maximize the mixed state density matrix over product states to find the entanglement measure because the resulting measure is not an entanglement monotone[46, 43, 92].

Eq.(2.7) connects directly the maximum probability of success with the reduced density operator

$$P_{\max}(\rho) = \max_{\varrho_1 \varrho_2 \cdots \widehat{\varrho_k} \cdots \varrho_n} \mathrm{tr}\left(\rho(\widehat{k})\varrho_1 \otimes \varrho_2 \otimes \ldots \widehat{\varrho_k} \ldots \otimes \varrho_n\right). \tag{2.8}$$

In fact, Theorem 1 is true for any entanglement measure. Consider an (n-1)-qudit reduced density matrix that can be purified by a single qudit reference system. Let $|\psi'\rangle$ be any joint pure state. All other purifications can be obtained from the state $|\psi'\rangle$ by LU-transformations $U \otimes \mathbb{1}^{\otimes(n-1)}$ where U is a local unitary matrix acting on single qudit and $\mathbb{1}$ is a unit matrix. Since any entanglement measure must be invariant under LU-transformations, it must be the same for all purifications independently of U. Hence the reduced density matrix ρ determines any entanglement measure on the initial pure state.

However, there is a crucial difference. In the case of Groverian measure the proof expresses entanglement measure by the reduced density matrix directly. As will be explained in Section IV, Eq.(2.8) is a simple and effective tool for calculating three-qubit entanglement measure. No such formula is known for other measures and general proof for other measures has limited practical significance.

Theorem 2. *If two pure n-qudit states have LU equivalent $(n-1)$-qudit reduced states, then they have equal Groverian and Geometric entanglement measures.*

Proof. Assume that the density matrices of pure states are ρ and ρ' and corresponding maximum probabilities of success are $P_{\max}$ and $P'_{\max}$. Suppose the local unitary transformation $U^1 \otimes U^2 \otimes \cdots \otimes U^{n-1}$ maps $\rho'(\widehat{k'}) = \mathrm{tr}_{k'}\rho'$ to $\rho(\widehat{k}) = \mathrm{tr}_k\rho$ as following:

$$\rho(\widehat{k}) = \left(U^1 \otimes U^2 \otimes \cdots \otimes U^{n-1}\right)\rho'(\widehat{k'})\left(U^1 \otimes U^2 \otimes \cdots \otimes U^{n-1}\right)^+, \tag{2.9}$$

where superscript + means hermitian conjugate. The trace with any complete product $\varrho^1 \otimes \varrho^2 \otimes \cdots \otimes \varrho^{n-1}$ state gives

$$\mathrm{tr}\left(\rho(\widehat{k})\varrho^1 \otimes \varrho^2 \otimes \cdots \otimes \varrho^{n-1}\right) = \mathrm{tr}\left(\rho'(\widehat{k'})\varrho'^1 \otimes \varrho'^2 \otimes \cdots \otimes \varrho'^{n-1}\right), \tag{2.10}$$

where $\varrho'^k = U^{k+}\varrho^k U^k$ are single qubit pure states too. Let's choose the product state that maximizes the l.h.s. According to Eq.(2.8) l.h.s is $P_{\max}$ and therefore $P_{\max} \leq P'_{\max}$. Similarly $P'_{\max} \leq P_{\max}$, therefore $P_{\max} = P'_{\max}$.

2.2 Lower bound for multi-qubit systems

Theorem 1 sets a clear lower bound for the maximum probability of success.

Below A is an arbitrary 2×2 hermitian matrix, r is a unit real three-dimensional vector and components of the vector σ are Pauli matrices. The trace of the product of matrices A and $r \cdot \sigma$ can be presented as a scalar product of vectors r and $\mathrm{tr}(A\sigma)$. The scalar product of two real vectors with the constant modules is maximal when vectors are parallel. Consequently, we have

$$\max_{r^2=1} \mathrm{tr}\,(A\, r\cdot\sigma) = |\mathrm{tr}(A\sigma)| = \sqrt{(\mathrm{tr}A)^2 - 4\det A} \tag{2.11}$$

and the positive root of radicals is understood.

An arbitrary density matrix ϱ for a pure state qubit may be written as $\varrho = 1/2\,(\mathbb{1} + r\cdot\sigma)$, where and r is a unit real vector. Then Eq.(2.11) can be rewritten as

$$\max_{\varrho} \mathrm{tr}\,(A\,\varrho) = \frac{1}{2}\left(\mathrm{tr}A + \sqrt{(\mathrm{tr}A)^2 - 4\det A}\right). \tag{2.12}$$

From Eq.(2.12) it follows that

$$\max_{\varrho} \mathrm{tr}\,(A\,\varrho) \geq \frac{1}{2}\,(\mathrm{tr}A)\,. \tag{2.13}$$

We define 2×2 matrix M_{n-1} by formula

$$M_{n-1} = \mathrm{tr}_{1,2,\ldots,n-2}\left(\rho(\widehat{n})\varrho^1 \otimes \varrho^2, \otimes \cdots \varrho^{n-2} \otimes \mathbb{1}\right). \tag{2.14}$$

where trace is taken over (1,2,...,n-2)-qubits. Eq.(2.8) takes the form

$$P_{\max} = \max_{\varrho^1\varrho^2\cdots\varrho^{n-1}} \mathrm{tr}(M_{n-1}\varrho^{n-1}), \tag{2.15}$$

where tr means trace over (n-1)-qubit. Eq.(2.13) gives

$$P_{\max} \geq \frac{1}{2} \max_{\varrho^1 \varrho^2 \cdots \varrho^{n-2}} \mathrm{tr} M_{n-1} = \frac{1}{2} \max_{\varrho^1 \varrho^2 \cdots \varrho^{n-2}} \mathrm{tr} \left(\rho(\widehat{n}) \varrho^1 \otimes \varrho^2 \otimes \cdots \varrho^{n-2} \otimes \mathbb{1} \right), \quad (2.16)$$

where tr in rhs of Eq.(2.16) means trace over all qubits.Thus inequality (2.13) suggests a simple prescription: replace a pure qubit density matrix by unit matrix and add a multiplier 1/2 instead. We use this prescription $n-1$ times, eliminate all single qubit density operators step by step from Eq.(2.8) and obtain

$$P_{\max} \geq \frac{1}{2^{n-1}}. \quad (2.17)$$

Note that this lower bound is valid only for pure states. The question at issue is whether it is a precise limit or not. And if it is indeed the case, then what are the pure states which have the lower bound of $P_{\max}$? We will prove that this lower bound is reached only for bipartite states.

Denote by $\rho^{k_1 k_2 \cdots k_m}$ the reduced density operator of qubits $k_1 k_2 \cdots k_m$, $1 \leq m \leq n-1$. Eq. (2.7) and (2.13) together yield

$$P_{\max}(\rho) \geq \frac{1}{2^{n-m-1}} P_{\max}(\rho^{k_1 k_2 \cdots k_m}). \quad (2.18)$$

Note, $P_{\max}(\rho^{k_1 k_2 \cdots k_m})$ does not define any entanglement measure as $\rho^{k_1 k_2 \cdots k_m}$'s are mixed states. It is the maximal overlap of the mixed state with any product state and we use it as intermediate mathematical quantity.

Lemma 2. *If a pure state has limiting Geometric / Groverian entanglement* $P_{\max} = 1/2^{n-1}$*, then all its reduced states are completely mixed states.*

Proof. Eq.(2.18) for $m = 1$ and Eq.(2.12) impose

$$P_{\max} \geq \frac{1}{2^{n-1}} \left(1 + \sqrt{1 - 4 \det \rho^k} \right). \quad (2.19)$$

The maximal probability of success reaches the minimal value if the square root vanishes. Consequently, density matrices ρ^k must be multiple of a unit matrix $\rho^k = \mathbb{1}/2$ and thus all one-qubit reduced states are completely mixed. Then two qubit density matrices $\rho^{k_1 k_2}$ must have the form

$$\rho^{k_1 k_2} = \frac{1}{4} \left(\mathbb{1} \otimes \mathbb{1} + g_{\alpha\beta} \, \sigma^\alpha \otimes \sigma^\beta \right). \quad (2.20)$$

where $g_{\alpha\beta} = \mathrm{tr}(\rho^{k_1 k_2} \sigma^\alpha \otimes \sigma^\beta)$ is a 3×3 matrix with real entries. Hereafter summation for repeated three dimensional vector indices $(\alpha, \beta, \gamma \cdots = 1, 2, 3)$ is understood unless otherwise stated. To reach the lower bound we must have equality instead of inequality in (2.18) and this condition imposes $P_{\max}(\rho^{k_1 k_2}) = 1/4$ resulting in $g_{\alpha\beta} = 0$. Hence $\rho^{k_1 k_2} = (1/4)\mathbb{1} \otimes \mathbb{1}$ and thus all two-qubit reduced states are completely mixed. One can continue this chain of derivations by induction. Indeed, suppose all m-qubit states ($m < n$) are completely mixed. Then $(m+1)$-qubit density matrices $\rho^{k_1 k_2 \cdots k_{m+1}}$ must have the form

$$\rho^{k_1 k_2 \cdots k_{m+1}} = \frac{1}{2^{m+1}} \left(\mathbb{1}^{\otimes m+1} + g_{\alpha_1 \alpha_2 \cdots \alpha_{m+1}} \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \otimes \sigma^{\alpha_{m+1}} \right), \tag{2.21}$$

where

$$g_{\alpha_1 \alpha_2 \cdots \alpha_{m+1}} = \mathrm{tr} \left(\rho^{k_1 k_2 \cdots k_{m+1}} \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \otimes \sigma^{\alpha_{m+1}} \right). \tag{2.22}$$

From Eq.(2.18) it follows that $P_{\max}(\psi)$ takes its minimal value if $P_{\max}(\rho^{k_1 k_2 \cdots k_m}) = 1/2^m$. Eq.(2.21) is consistent with this condition if and only if the maximization of the term of $g_{\alpha_1 \alpha_2 \cdots \alpha_{m+1}} \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \sigma^{\alpha_{m+1}}$ yields zero. Then $g_{\alpha_1 \alpha_2 \cdots \alpha_{m+1}} = 0$ and therefore

$$\rho^{k_1 k_2 \cdots k_{m+1}} = \frac{1}{2^{m+1}} \mathbb{1}^{\otimes m+1}. \tag{2.23}$$

Thus if all m-qubit reduced states are completely mixed then all $(m+1)$-qubit reduced states are also completely mixed. On the other hand all one-qubit reduced state are completely mixed. By induction all reduced states are completely mixed. The induction stops at pure states. In contrast to mixed states, the maximization of the term $g_{\alpha_1 \alpha_2 \cdots \alpha_n} \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \sigma^{\alpha_n}$ must yield unity for pure states as requires Eq.(2.7).

Lemma is proved.

Theorem 3. *None of multi-qubit pure states except two-qubit maximally entangled states satisfies the condition* $P_{\max} = 1/2^{n-1}$.

Proof. When $n = 2$, it is well-known that the EPR states and their LU-equivalent class reach the lower bound, *i.e.* $P_{max} = 1/2$. Now we would like to show that there is no pure state with limiting Groverian measure for $n = 3$. Lemma 2 requires that the density matrix with limiting Groverian measure should be in the form

$$\rho = \frac{1}{8} \left(\mathbb{1}^{\otimes 3} + g_{\alpha\beta\gamma} \sigma^{\alpha} \otimes \sigma^{\beta} \otimes \sigma^{\gamma} \right). \tag{2.24}$$

Since ρ is a pure state density matrix, it must satisfy $\rho^2 = \rho$. This condition leads several constraints, one of which is

$$-i g_{\alpha\beta\gamma} g_{\delta\kappa\lambda} \epsilon_{\alpha\delta\delta'} \epsilon_{\beta\kappa\kappa'} \epsilon_{\gamma\lambda\lambda'} \sigma^{\delta'} \otimes \sigma^{\kappa'} \otimes \sigma^{\lambda'} = 6 g_{\alpha\beta\gamma} \sigma^{\alpha} \otimes \sigma^{\beta} \otimes \sigma^{\gamma} \tag{2.25}$$

where $\epsilon_{\alpha\beta\gamma}$ is an antisymmetric tensor. Since this constraint cannot be satisfied for real $g_{\alpha\beta\gamma}$, there is no pure state which has limiting Groverian measure at $n = 3$.

Now we will show that there is no pure state for $n \geq 4$ too. Suppose there is n-qubit state $|\psi\rangle$ such that all its reduced states are completely mixed. Choose a normalized basis of product vectors $|i_1 i_2 \cdots i_n\rangle$ where the labels within ket refer to qubits $1, 2, \cdots n$ in that order. The vector $|\psi\rangle$ can be written as a linear combination

$$|\psi\rangle = \sum_{i_1 i_2 \cdots i_n} C_{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle \tag{2.26}$$

of vectors in the set. All reduced states of the state $|\psi\rangle$ are completely mixed if and only if

$$\sum_{i_k j_k} \delta_{i_k j_k} C_{i_1 i_2 \cdots i_n} C^*_{j_1 j_2 \cdots j_n} = \frac{1}{2^{n-1}} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \widehat{\delta_{i_k j_k}} \cdots \delta_{i_n j_n}, \quad k = 1, 2, \cdots n. \tag{2.27}$$

Note that normalization condition follows from above equation. Define $n - 1$ index coefficients

$$D_{i_1 i_2 \cdots i_{n-1}} = \sqrt{2} C_{i_1 i_2 \cdots i_{n-1} 0}. \tag{2.28}$$

Setting $i_n = j_n = 0$ in Eq.(2.27) we get

$$\sum_{i_k j_k} \delta_{i_k j_k} D_{i_1 i_2 \cdots i_{n-1}} D^*_{j_1 j_2 \cdots j_{n-1}} = \frac{1}{2^{n-2}} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \widehat{\delta_{i_k j_k}} \cdots \delta_{i_{n-1} j_{n-1}}, \ k = 1, 2, \cdots n-1. \tag{2.29}$$

Hence the $(n - 1)$-qubit state

$$|\phi\rangle = \sum_{i_1 i_2 \cdots i_{n-1}} D_{i_1 i_2 \cdots i_{n-1}} |i_1 i_2 \cdots i_{n-1}\rangle \tag{2.30}$$

exists and all its reduced states are completely mixed. The contraposition of it is that if there is no pure state which has limiting Groverian measure at $n = 3$, it is also true for $n \geq 4$. Theorem 3 is proved.

Thus, the lower bound of inequality (2.17) is unreachable for $n \geq 3$. This seems to mean that Eq.(2.17) is not a precise limit.

2.3 Analytic expressions for maximum probability of success

The maximization of the pure three qubit states over product states generally reduces to *nonlinear* eigenvalue equations [43]. However, Eq.(2.8) converts it effectively into *linear* eigenvalue equations. Thus, one can compute the entanglement measures for wide range of three qubit states analytically. As an illustration consider one parametric W-type [65] three qubit state

$$|\psi\rangle = \frac{1}{\sqrt{1+\kappa^2+\kappa^4}}(|100\rangle + \kappa|010\rangle + \kappa^2|001\rangle), \tag{2.31}$$

where κ is a free positive parameter. The calculation method is elaborated in Ref.[66] and here we present only final results. In three different ranges of definition the maximal success probability is differently expressed. In the first case $P_{\max}$ is the square of the first coefficient provided it is greater than 1/2:

$$P_{\max} = \frac{1}{1+\kappa^2+\kappa^4}, \quad 0 < \kappa < \left(\frac{\sqrt{5}-1}{2}\right)^{1/2}. \tag{2.32}$$

In the second case $P_{\max}$ is the square of the diameter of the circumcircle of the acute triangle formed by three coefficients:

$$P_{\max} = \frac{4\kappa^6}{(1+\kappa^2+\kappa^4)^2(3\kappa^2-1-\kappa^4)}, \quad \left(\frac{\sqrt{5}-1}{2}\right)^{1/2} \leq \kappa \leq \left(\frac{\sqrt{5}+1}{2}\right)^{1/2}. \tag{2.33}$$

In the third case $P_{\max}$ is the square of the third coefficient provided it is greater than 1/2:

$$P_{\max} = \frac{\kappa^4}{1+\kappa^2+\kappa^4}, \quad \kappa > \left(\frac{\sqrt{5}+1}{2}\right)^{1/2}. \tag{2.34}$$

It is also possible to compute $P_{\max}$ for Eq.(2.31) numerically[93]. For numerical calculation we consider k^{th} qubit as $|q_k\rangle = \cos\theta_k|0\rangle + e^{i\varphi_k}\sin\theta_k|1\rangle$ with $k = 1, 2, 3$. Since the coefficients of $|\psi\rangle$ are all real, we can put $\varphi_k = 0$ for all k and express $P_{\max}$ in a form

$$P_{max} = \max_{\theta_1,\theta_2,\theta_3} |\langle q_1|\langle q_2|\langle q_3|\psi\rangle|^2. \tag{2.35}$$

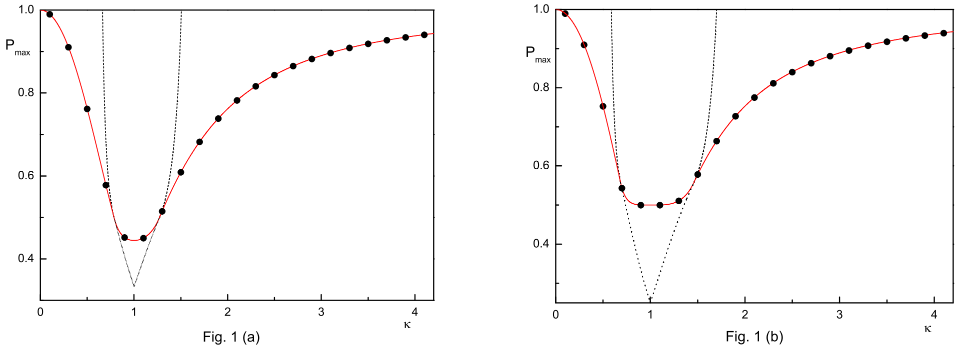

Figure 2.1: P_{max} for Eq.(2.31) (Fig. 1 a) and Eq.(2.36) (Fig. 1 b). The solid lines represent the analytical results of P_{max} and the black dots are the numerical results. This figures strongly support that our analytical results are perfect correct.

Thus numerical maximization over θ_1, θ_2 and θ_3 directly yields $P_{\max}$. As shown in Fig. 1(a) the numerical result (black dots) perfectly coincides with the analytic results (solid lines) expresses in Eq.(2.32), (2.33) and (2.34).

Let us consider another one parametric state

$$|\psi\rangle = \frac{1}{\sqrt{1+\kappa^2+\kappa^4+\kappa^6}} \left(|100\rangle + \kappa|010\rangle + \kappa^2|001\rangle + \kappa^3|111\rangle\right). \tag{2.36}$$

Again there are three cases. If four coefficients form a cyclic quadrilateral, then $P_{\max} = 4R^2$, where R is the circumradius of the quadrangle. Otherwise $P_{\max}$ is the square of the largest coefficient. In the first case $P_{\max}$ is the square of first coefficient:

$$\begin{aligned} P_{\max} &= \frac{1}{1+\kappa^2+\kappa^4+\kappa^6}, \\ \kappa &< \frac{1}{3}\left(\sqrt[3]{18\sqrt{57}+134} - \sqrt[3]{18\sqrt{57}-134} - 1\right)^{1/2} \approx 0.685. \end{aligned} \tag{2.37}$$

In the second case $P_{\max}$ is the square of the circumcircle of the cyclic quadrangle formed by four coefficients:

$$
\begin{aligned}
&P_{\max} = \frac{8\kappa^6}{-1+2\kappa^2+\kappa^4+8\kappa^6+\kappa^8+2\kappa^{10}-\kappa^{12}}, \\
&\frac{1}{3}\left(\sqrt[3]{18\sqrt{57}+134}-\sqrt[3]{18\sqrt{57}-134}-1\right)^{1/2} \leq \kappa \leq \frac{1}{\sqrt{3}}\left(\sqrt[3]{46+6\sqrt{57}}+\sqrt[3]{46-6\sqrt{57}}+1\right)^{1/2}.
\end{aligned}
\tag{2.38}
$$

In the third case $P_{\max}$ is the square of the last coefficient:

$$
\begin{aligned}
&P_{\max} = \frac{\kappa^6}{1+\kappa^2+\kappa^4+\kappa^6}, \\
&\kappa > \frac{1}{\sqrt{3}}\left(\sqrt[3]{46+6\sqrt{57}}+\sqrt[3]{46-6\sqrt{57}}+1\right)^{1/2} \approx 1.46.
\end{aligned}
\tag{2.39}
$$

The function $P_{\max}(k)$ and numerical results are shown in Fig. 1(b). Both figures strongly show that our analytical expressions of $P_{\max}$ perfectly coincide with the numerical result.

2.4 Conclusions

Eq.(2.8) allows to calculate the maximal success probability for three qubit states which are expressed as linear combinations of four given orthogonal product states [67]. The answer is more complicated than a simple formula, but each final expression of the measure has its own meaningful interpretation. Namely, $P_{\max}$ can take the following values(up to numerical coefficients):

- the square of the circumradius of the cyclic polygon formed by coefficients of the state function,
- the square of the circumradius of the crossed figure formed by coefficients of the state function,
- the largest coefficient.

Each expression has its own range of definition where they are applicable. Although the above picture seems simple, the separation of the applicable domains is highly nontrivial task. To make clear which of expressions should be applied for a given state we refer to [67]. All our results on Groverian measure of three qubit pure states are summarized in [69].

Eq.(2.8) gives nonlinear eigenvalue problem for four and higher qubit states and it is natural to ask whether there is an extension of Eq.(2.8) that allows to find analytic results for four, five, or general n-qubits. Although we have no distinct results here, but we have obtained some insight from the analysis of the information contained in one and two qubit reduced states. Probably, it is possible to express the maximal success probability in terms of one and two qubit reduced states in case of four qubit pure states. Such formula, if it can be derived, will give linear equations for four qubit pure states. However, situation is opposite in the case of five qubit states. The method does now allow to convert the task to the linear eigenvalue problem and more powerful tools are needed to calculate maximal success probability of general n-qubit states.

Chapter 3

Analytic expressions for geometric measure of three-qubit states

Entangled states have different remarkable applications and among them are quantum cryptography [21, 94], superdense coding [20, 95], teleportation [19, 96] and the potential speedup of quantum algorithms [97, 32, 98]. The entanglement of bipartite systems is well-understood [37, 36, 39, 86], while the entanglement of multipartite systems offers a real challenge to physicists. In contrast to bipartite setting, there is no unique treatment of the maximally entangled states for multipartite systems. In this reason it is highly difficult to formulate a theory of multipartite entanglement. Another point which makes difficult to understand the entanglement for the multi-qubit systems is mainly due to the fact that the analytic expressions for the various entanglement measures is extremely hard to derive.

We consider pure three qubit systems [68, 99, 71, 87], although the entanglement of mixed states attracts a considerable attention. For example, in recent experiment [100] the tangle for general mixed states was evaluated, which has never been done before. Three-qubit system is important in the sense that it is the simplest system which gives a non-trivial effect in the entanglement. Thus, we should understand the general properties of the entanglement in this system as much as possible to go further more complicated higher qubit system. The three-qubit system can be entangled in two inequivalent ways GHZ [101] and W, and neither form can be transformed into the other with any probability of success [65]. This picture is complete: any fully entangled state is SLOCC equivalent to either GHZ or W.

Only very few analytical results for tripartite entanglement have been obtained so far [102] and we need more light on the subject. This is our main objective and we choose geometric measure of entanglement E_g [33, 41, 42, 43]. It is an axiomatic measure [33, 89, 90, 91], is connected with other measures [103, 104] and has an operational treatment. Namely, for the case of pure states it is closely related to the Groverian measure of entanglement [46] and the latter is associated with the success probability of Grover's search algorithm [24] when a given state is used as the initial state.

Geometric measure depends on entanglement eigenvalue $\Lambda^2_{\max}$ and is given by formula $E_g(\psi) = 1-\Lambda^2_{\max}$. For pure states the entanglement eigenvalue is equal to the maximal overlap of a given state with any complete product state. The maximization over product states gives *nonlinear eigenproblem* [43] which, except rare cases, does not allow the complete analytical solutions.

Recently the idea was suggested that nonlinear eigenproblem can be reduced to the linear eigenproblem for the case of three qubit pure states [64]. The idea is based on theorem stating that any reduced $(n-1)$-qubit state uniquely determines the geometric measure of the original n-qubit pure state. This means that two qubit mixed states can be used to calculate the geometric measure of three qubit pure states and this will be fully addressed in this chapter.

The method gives two algebraic equations of degree six defining the geometric measure of entanglement. Thus the difficult problem of geometric measure calculation is reduced to the algebraic equation root finding. Equations contain valuable information, are good bases for the numerical calculations and may test numerical calculations based on other numerical techniques [98].

Furthermore, the method allows to find the nearest separable states for three qubit states of most interest and get analytic expressions for their geometric measures. It turn out that highly entangled states have their own feature. Each highly entangled state has a vicinity with no product state and all nearest product states are on the boundary of the vicinity and form an one-parametric set.

In Section 3.1 we derive algebraic equations defining the geometric entanglement measure of pure three qubit states and present the general solution. In Section 3.2 we examine W-type states and deduce analytic expression for their geometric measures. States symmetric under permutation of two qubits are considered in Section 3.3, where the overlap of the state functions with the product states are maximized directly. In last Section 3.4 we make concluding remarks of this chapter.

3.1 Algebraic equations.

We consider three qubits A,B,C with state function $|\psi\rangle$. The entanglement eigenvalue is given by

$$\Lambda_{max} = \max_{q^1 q^2 q^3} |\langle q^1 q^2 q^3 | \psi \rangle| \tag{3.1}$$

and the maximization runs over all normalized complete product states $|q^1\rangle \otimes |q^2\rangle \otimes |q^3\rangle$. Superscripts label single qubit states and spin indices are omitted for simplicity. Since in the following we will use density matrices rather than state functions, our first aim is to rewrite Eq.(3.1) in terms of density matrices. Let us denote by $\rho^{ABC} = |\psi\rangle\langle\psi|$ the density matrix of the three-qubit state and by $\varrho^k = |q^k\rangle\langle q^k|$ the density matrices of the single qubit states. The equation for the square of the entanglement eigenvalue takes the form

$$\Lambda^2_{\max}(\psi) = \max_{\varrho^1 \varrho^2 \varrho^3} \mathrm{tr}\left(\rho^{ABC} \varrho^1 \otimes \varrho^2 \otimes \varrho^3\right). \tag{3.2}$$

An important equality

$$\max_{\varrho^3} \mathrm{tr}(\rho^{ABC} \varrho^1 \otimes \varrho^2 \otimes \varrho^3) = \mathrm{tr}(\rho^{ABC} \varrho^1 \otimes \varrho^2 \otimes \mathbb{1}^3) \tag{3.3}$$

was derived in [64] where $\mathbb{1}$ is a unit matrix. It has a clear meaning. The matrix $\mathrm{tr}(\rho^{ABC} \varrho^1 \otimes \varrho^2)$ is $2 \otimes 2$ hermitian matrix and has two eigenvalues. One of eigenvalues is always zero and another is always positive and therefore the maximization of the matrix simply takes the nonzero eigenvalue. Note that its minimization gives zero as the minimization takes the zero eigenvalue.

We use Eq.(3.3) to reexpress the entanglement eigenvalue by reduced density matrix ρ^{AB} of qubits A and B in a form

$$\Lambda^2_{\max}(\psi) = \max_{\varrho^1 \varrho^2} \mathrm{tr}\left(\rho^{AB} \varrho^1 \otimes \varrho^2\right). \tag{3.4}$$

We denote by s_1 and s_2 the unit Bloch vectors of the density matrices ϱ^1 and ϱ^2 respectively and adopt the usual summation convention on repeated indices i and j. Then

$$\Lambda^2_{\max} = \frac{1}{4} \max_{s_1^2 = s_2^2 = 1} \left(1 + s_1 \cdot r_1 + s_2 \cdot r_2 + g_{ij}\, s_{1i} s_{2j}\right), \tag{3.5}$$

where

$$r_1 = \mathrm{tr}(\rho^A \sigma), \; r_2 = \mathrm{tr}(\rho^B \sigma), \; g_{ij} = \mathrm{tr}(\rho^{AB} \sigma_i \otimes \sigma_j) \tag{3.6}$$

and σ_i's are Pauli matrices. The matrix g_{ij} is not necessarily to be symmetric but must has only real entries. The maximization gives a pair of equations

$$r_1 + g s_2 = \lambda_1 s_1, \quad r_2 + g^T s_1 = \lambda_2 s_2, \tag{3.7}$$

where Lagrange multipliers λ_1 and λ_2 are enforcing unit nature of the Bloch vectors. The solution of Eq.(3.7) is

$$s_1 = \left(\lambda_1 \lambda_2 \mathbb{1} - g\, g^T\right)^{-1} \left(\lambda_2 r_1 + g\, r_2\right), \tag{3.8a}$$

$$s_2 = \left(\lambda_1 \lambda_2 \mathbb{1} - g^T g\right)^{-1} \left(\lambda_1 r_2 + g^T r_1\right). \tag{3.8b}$$

Now, the only unknowns are Lagrange multipliers, which should be determined by equations

$$|s_1|^2 = 1, \quad |s_2|^2 = 1. \tag{3.9}$$

In general, Eq.(3.9) give two algebraic equations of degree six. However, the solution (3.8) is valid if Eq.(3.7) supports a unique solution and this is by no means always the case. If the solution of Eq.(3.7) contains a free parameter, then $\det(\lambda_1 \lambda_2 \mathbb{1} - g g^T) = 0$ and, as a result, Eq.(3.8) cannot not applicable. The example presented in Section III will demonstrate this situation.

In order to test Eq.(3.8) let us consider an arbitrary superposition of W

$$|W\rangle = \frac{1}{\sqrt{3}} \left(|100\rangle + |010\rangle + |001\rangle\right) \tag{3.10}$$

and flipped W

$$|\widetilde{W}\rangle = \frac{1}{\sqrt{3}} \left(|011\rangle + |101\rangle + |110\rangle\right) \tag{3.11}$$

states, i.e. the state

$$|\psi\rangle = \cos\theta\, |W\rangle + \sin\theta\, |\widetilde{W}\rangle. \tag{3.12}$$

Straightforward calculation yields

$$r_1 = r_2 = \frac{1}{3} \left(2 \sin 2\theta\, i + \cos 2\theta\, n\right), \tag{3.13a}$$

$$g = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.13b}$$

where unit vectors i and n are aligned with the axes x and z, respectively. Both vectors i and n are eigenvectors of matrices g and g^T. Therefore s_1 and s_2 are linear combinations of i and n. Also from $r_1 = r_2$ and $g = g^T$ it follows that $s_1 = s_2$ and $\lambda_1 = \lambda_2$. Then Eq.(3.8) for general solution give

$$s_1 = s_2 = \sin 2\varphi\, i + \cos 2\varphi\, n \tag{3.14}$$

where

$$\sin 2\varphi = \frac{2\sin 2\theta}{3\lambda - 2}, \quad \cos 2\varphi = \frac{\cos 2\theta}{3\lambda + 1}. \tag{3.15}$$

The elimination of the Lagrange multiplier λ from Eq.(3.15) gives

$$3\sin 2\varphi \cos 2\varphi = \cos 2\theta \sin 2\varphi - 2\sin 2\theta \cos 2\varphi. \tag{3.16}$$

Let us denote by $t = \tan\varphi$. After the separation of the irrelevant root $t = -\tan\theta$, Eq.(3.16) takes the form

$$\sin\theta\, t^3 + 2\cos\theta\, t^2 - 2\sin\theta\, t - \cos\theta = 0. \tag{3.17}$$

This equation exactly coincides with that derived in [43]. Since a detailed analysis was given in Ref.[43], we do not want to repeat the same calculation here. Instead we would like to consider the three-qubit states that allow the analytic expressions for the geometric entanglement measure by making use of Eq.(3.7).

3.2 W-type states.

Consider W-type state

$$|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle, \quad a^2 + b^2 + c^2 = 1. \tag{3.18}$$

Without loss of generality we consider only the case of positive parameters a, b, c. Direct calculation yields

$$r_1 = r_1\, n, \quad r_2 = r_2\, n, \quad g = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & -r_3 \end{pmatrix}, \tag{3.19}$$

where

$$r_1 = b^2 + c^2 - a^2,\ r_2 = a^2 + c^2 - b^2,\ r_3 = a^2 + b^2 - c^2 \tag{3.20}$$

and $\omega = 2ab$. The unit vector n is aligned with the axis z. Any vector perpendicular to n is an eigenvector of g with eigenvalue ω. Then from Eq.(3.7) it follows that the components of vectors s_1 and s_2 perpendicular to n are collinear. We denote by m the unit vector along that direction and parameterize vectors s_1 and s_2 as follows

$$s_1 = \cos\alpha\, n + \sin\alpha\, m, \quad s_2 = \cos\beta\, n + \sin\beta\, m. \tag{3.21}$$

Then Eq.(3.7) reduces to the following four equations

$$r_1 - r_3\cos\beta = \lambda_1\cos\alpha, \quad r_2 - r_3\cos\alpha = \lambda_2\cos\beta, \tag{3.22a}$$

$$\omega\sin\beta = \lambda_1\sin\alpha, \quad \omega\sin\alpha = \lambda_2\sin\beta, \tag{3.22b}$$

which are used to solve the four unknown constants $\lambda_1, \lambda_2, \alpha$ and β. Eq.(3.22b) impose either

$$\lambda_1\lambda_2 - \omega^2 = 0 \tag{3.23}$$

or

$$\sin\alpha\sin\beta = 0. \tag{3.24}$$

First consider the case $r_1 > 0$, $r_2 > 0$, $r_3 > 0$ and coefficients a, b, c form an acute triangle. Eq.(3.24) does not give a true maximum and this can be understood as follows. If both vectors s_1 and s_2 are aligned with the axis z, then the last term in Eq.(3.5) is negative. If vectors s_1 and s_2 are antiparallel, then one of scalar products in Eq.(3.5) is negative. In this reason $\Lambda^2_{\max}$ cannot be maximal. Then Eq.(3.23) gives true maximum and we have to choose positive values for λ_1 and λ_2 to get maximum.

First we use Eq.(3.22a) to connect the angles α and β with the Lagrange multipliers λ_1 and λ_2

$$\cos\alpha = \frac{\lambda_2 r_1 - r_2 r_3}{\omega^2 - r_3^2}, \quad \cos\beta = \frac{\lambda_1 r_2 - r_1 r_3}{\omega^2 - r_3^2}. \tag{3.25}$$

Then Eq.(3.22b) and (3.23) give the following expressions for Lagrange multipliers λ_1 and λ_2

$$\lambda_1 = \omega\left(\frac{\omega^2 + r_1^2 - r_3^2}{\omega^2 + r_2^2 - r_3^2}\right)^{1/2}, \tag{3.26a}$$

$$\lambda_2 = \omega \left(\frac{\omega^2 + r_2^2 - r_3^2}{\omega^2 + r_1^2 - r_3^2} \right)^{1/2} . \tag{3.26b}$$

Eq.(3.7) allows to write a shorter expression for the entanglement eigenvalue

$$\Lambda_{\max}^2 = \frac{1}{4} \left(1 + \lambda_2 + r_1 \cos\alpha \right) . \tag{3.27}$$

Now we insert the values of λ_2 and $\cos\alpha$ into Eq.(3.27) and obtain

$$4\Lambda_{\max}^2 = 1 + \frac{\omega\sqrt{(\omega^2 + r_1^2 - r_3^2)(\omega^2 + r_2^2 - r_3^2)} - r_1 r_2 r_3}{\omega^2 - r_3^2}. \tag{3.28}$$

The denominator in above expression is multiple of the area S of the triangle a, b, c

$$\omega^2 - r_3^2 = 16S^2. \tag{3.29}$$

A little algebra yields for the numerator

$$\begin{aligned} &\omega\sqrt{(\omega^2 + r_1^2 - r_3^2) + (\omega^2 + r_2^2 - r_3^2)} - r_1 r_2 r_3 \\ &= 16\, a^2 b^2 c^2 - \omega^2 + r_3^2. \end{aligned} \tag{3.30}$$

Combining together the numerator and denominator, we obtain the final expression for the entanglement eigenvalue

$$\Lambda_{\max}^2 = 4R^2, \tag{3.31}$$

where R is the circumradius of the triangle a, b, c. Entanglement value is minimal when triangle is regular, i.e. for W-state and $\Lambda_{\max}^2(W) = 4/9$ [93, 43].

Now consider the case $r_3 < 0$. Since $r_3 + r_1 = 2b^2 \geq 0$, we have $r_1 > 0$ and similarly $r_2 > 0$. Eq.(3.24) gives true maximum in this case and both vectors are aligned with the axis z

$$s_1 = s_2 = n \tag{3.32}$$

resulting in $\Lambda_{\max}^2 = c^2$. In view of symmetry

$$\Lambda_{\max}^2 = \max(a^2, b^2, c^2), \quad \max(a^2, b^2, c^2) > \frac{1}{2}. \tag{3.33}$$

Since the matrix g and vectors r_1 and r_2 are invariant under rotations around axis z the same properties must have Bloch vectors s_1 and s_2. There are two possibilities:

i)Bloch vectors are unique and aligned with the axis z. The solution given by Eq.(3.32) corresponds to this situation and the resulting entanglement eigenvalue Eq.(3.33) satisfies the inequality

$$\frac{1}{2} < \Lambda^2_{\max} \leq 1. \tag{3.34}$$

ii)Bloch vectors have nonzero components in xy plane and the solution is not unique. Eq.(3.21) corresponds to this situation and contains a free parameter. The free parameter is the angle defining the direction of the vector m in the xy plane. Then Eq.(3.31) gives the entanglement eigenvalue in highly entangled region

$$\frac{4}{9} \leq \Lambda^2_{\max} < \frac{1}{2}. \tag{3.35}$$

Eq.(3.31) and (3.33) have joint curves when parameters a, b, c form a right triangle and give $\Lambda^2_{\max} = 1/2$. The GHZ states have same entanglement value and it seems to imply something interesting. GHZ state can be used for teleportation and superdense coding, but W-state cannot be. However, the W-type state with right triangle coefficients can be used for teleportation and superdense coding [105]. In other words, both type of states can be applied provided they have the required entanglement eigenvalue $\Lambda^2_{\max} = 1/2$.

3.3 Symmetric States.

Now let us consider the state which is symmetric under permutation of qubits A and B and contains three real independent parameters

$$|\psi\rangle = a|000\rangle + b|111\rangle + c|001\rangle + d|110\rangle, \tag{3.36}$$

where $a^2 + b^2 + c^2 + d^2 = 1$. According to Generalized Schmidt Decomposition [68] the states with different sets of parameters are local-unitary(LU) inequivalent. The relevant quantities are

$$r_1 = r_2 = r\,n, \quad g = \begin{pmatrix} \omega & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.37}$$

where

$$r = a^2 + c^2 - b^2 - d^2, \quad \omega = 2ad + 2bc \tag{3.38}$$

and the unit vector n again is aligned with the axis z.

All three terms in the l.h.s. of Eq.(3.5) are bounded above:

- $s_1 \cdot r_1 \le |r|$,
- $s_2 \cdot r_2 \le |r|$,
- and owing to inequality $|\omega| \le 1$, $g_{ij}\, s_{1i} s_{2j} \le 1$.

Quite surprisingly all upper limits are reached simultaneously at

$$s_1 = s_2 = \mathrm{Sign}(r) n, \tag{3.39}$$

which results in

$$\Lambda^2_{\max} = \frac{1}{2}\left(1 + |r|\right). \tag{3.40}$$

This expression has a clear meaning. To understand it we parameterize the state as

$$|\psi\rangle = k_1 |00q_1\rangle + k_2 |11q_2\rangle, \tag{3.41}$$

where q_1 and q_2 are arbitrary single normalized qubit states and positive parameters k_1 and k_2 satisfy $k_1^2 + k_2^2 = 1$. Then

$$\Lambda^2_{\max} = \max(k_1^2, k_2^2), \tag{3.42}$$

i.e. the maximization takes a larger coefficient in Eq.(3.41). In bipartite case the maximization takes the largest coefficient in Schmidt decomposition [46, 106] and in this sense Eq.(3.41) effectively takes the place of Schmidt decomposition. When $|q_1\rangle = |0\rangle$ and $|q_2\rangle = |1\rangle$, Eq.(3.42) gives the known answer for generalized GHZ state [93, 43].

The entanglement eigenvalue is minimal $\Lambda^2_{\max} = 1/2$ on condition that $k_1 = k_2$. These states can be described as follows

$$|\psi\rangle = |00q_1\rangle + |11q_2\rangle \tag{3.43}$$

where q_1 and q_2 are arbitrary single qubit normalized states. The entanglement eigenvalue is constant $\Lambda^2_{\max} = 1/2$ and does not depend on single qubit state parameters. Hence one may expect that all these states can be applied for teleportation and superdense coding. It would be interesting to check whether this assumption is correct or not.

It turns out that GHZ state is not a unique state and is one of two-parametric LU inequivalent states that have $\Lambda^2_{\max} = 1/2$. On the other hand W-state is unique up to LU transformations and the low bound $\Lambda^2_{\max} = 4/9$ is reached if and only if $a = b = c$. However, one cannot make such conclusions in general. Five real parameters are necessary to parameterize the set of inequivalent three qubit pure states [68]. And there is no explicit argument that W-state is not just one of LU inequivalent states that have $\Lambda^2_{\max} = 4/9$.

3.4 Summary.

We have derived algebraic equations defining geometric measure of three qubit pure states. These equations have a degree higher than four and explicit solutions for general cases cannot be derived analytically. However, the explicit expressions are not important. Remember that explicit expressions for the algebraic equations of degree three and four have a limited practical significance but the equations itself are more important. This is especially true for equations of higher degree; main results can be derived from the equations rather than from the expressions of their roots.

Eq.(3.7) give the nearest separable state directly and this separable states have useful applications. In order to construct an entanglement witness, for example, the crucial point lies in finding the nearest separable state [107]. This will be especially interesting for highly entangled states that have a whole set of nearest separable states and allow to construct a set of entanglement witnesses.

The expression in r.h.s. of Eq.(3.5) can be maximized directly for various three qubit states. Although it is very hard to solve the higher-degree equation, it turns out that the wide range of the three-qubit states have a symmetry and this symmetry reduces the equations of degree six to the quadratic equations. In this reason Eq.(3.5) can be used to derive the analytic expressions of the various entanglement measures for the three-qubit states. Also Eq.(3.5) can be a starting point to explore the numerical computation of the entanglement measures for the higher-qubit systems. We would like to discuss this issue elsewhere.

Chapter 4

Geometric measure of entanglement and shared quantum states

Entanglement is the most intriguing feature of quantum mechanics and a key resource in quantum information science. One of the main goals in these theories is to develop a comprehensive theory of multipartite entanglement. Various entanglement measures have been invented to quantify the multi-particle entanglement [37, 36, 33, 35, 108, 109, 43] but none of them were able to suggest a method for calculating a measure of multipartite systems. This mathematical difficulty is the main obstacle to elaborate a theory of multi-particle entanglement.

In this chapter, we present the first calculation of the geometric measure of entanglement [43, 41, 42] for three qubit states which are expressed as linear combinations of four given orthogonal product states. Any pure three qubit state can be written in terms of five preassigned orthogonal product states [68] via Schmidt decomposition. Thus the states discussed here are more general states compared to the well-known GHZ [110] and W [65] states.

The reason for using the geometric measure of entanglement is that it is suitable for any partite system regardless of its dimensions. However, analytical computation for generic states still remains as a great challenge. The measure depends on entanglement eigenvalue $\Lambda^2_{\max}$ and can be derived from the formula $E_g(\psi) = 1 - \Lambda^2_{\max}$. For pure states, the entanglement eigenvalue is equal to the maximal overlap of a given state with any complete product state. This measure has the following remarkable properties:

i) it has an operational treatment. The same overlap $\Lambda^2_{\max}$ defines Groverian measure of entanglement [46, 92] which has been introduced later in operational terms. In other words, it quantifies how well a given state serves as an input state to Grover's search algorithm [24]. From this view, Groverian measure can be regarded as an operational treatment of the geometric measure.

ii) it has identified irregularity in channel capacity additivity [111]. Using this measure, one can show that a family of quantities, which were thought to be additive in an earlier papers, actually are not. For example, it is natural to conjecture that preparing two pairs of entangled particles should give us twice the entanglement of one pair and, similarly, using a channel twice doubles its capacity. However, this conjecture claiming additivity has proved to be wrong in some cases.

iii) it has useful connections to other entanglement measures and gives rise to a lower bound on the relative entropy of entanglement [103] and generalized robustness [104]. For certain pure states the first lower bound is saturated and thus their relative entropy of entanglement can be deduced from their geometric measure of entanglement. The second lower bound to generalized robustness can be express in terms of $\Lambda^2_{\max}$ directly.

Owing to these features, the geometric measure can play an important role in the investigation of different problems related to entanglement. For example, the entanglement of two distinct multipartite bound entangled states can be determined analytically in terms of a geometric measure of entanglement [112]. Recently, the same measure has been used to understand the physical implication of Zamolodchikov's c-theorem [47] more deeply. It is an important application regarding the quantum information techniques in the effect of renormalization group in field theories [48]. Thus it is natural that geometric measure of entanglement is an object of intense interest and in some recent works revised [113] and generalized [114] versions of the geometric measure were presented.

The progress made to date allows oneself to calculate the geometric measure of entanglement for pure three qubit systems [64]. The basic idea is to use $(n-1)$-qubit mixed states to calculate the geometric measure of n-qubit pure states. In the case of three qubits this idea converts the task effectively into the maximization of the two-qubit mixed state over product states and yields linear eigenvalue equations [66]. The solution of these linear eigenvalue equations reduces to the root finding for algebraic equations of degree six. However, three-qubit states containing symmetries allow complete analytical solutions and explicit expressions as the symmetry reduces the equations of degree six to the quadratic equations. Analytic expressions derived in this way are unique and the presented effective method can be applied for extended quantum systems. Our aim is to derive analytic expressions for a wider class of three qubit systems and in this sense this chapter is the continuation of Ref.[66].

We consider most general W-type three qubit states that allow to derive analytic expressions for entanglement eigenvalue. These states can be expressed as linear combinations of four given orthogonal product states. If any of coefficients in this expansion vanishes, then one obtains the states analyzed in [66]. Notice that arbitrary linear combinations of five product states [68] give a couple of algebraic equations of degree six. Hence Évariste Galois's theorem does not allow to get analytic expressions for these states except some particular cases.

We derive analytic expressions for an entanglement eigenvalue. Each expression has its own applicable domain depending on state parameters and these applicable domains are split up by separating surfaces. Thus the geometric measure distinguishes different types of states depending on the corresponding applicable domain. States that lie on separating surfaces are shared by two types of states and acquire new features.

In Section 4.1 we derive stationarity equations and their solutions. In Section 4.2 we specify three qubit states under consideration and find relevant quantities. In Section 4.3 we calculate entanglement eigenvalues and present explicit expressions. In Section 4.4 we separate the validity domains of the derived expressions. In Section 4.5 we discuss shared states. In section 4.6 we make concluding remarks of this chapter.

4.1 Stationarity equations

In this section we briefly review the derivation of the stationarity equations and their general solutions [66]. Denote by ρ^{ABC} the density matrix of the three-qubit pure state and define the entanglement eigenvalue $\Lambda^2_{\max}$ [43]

$$\Lambda^2_{\max} = \max_{\varrho^1 \varrho^2 \varrho^3} \operatorname{tr}\left(\rho^{ABC} \varrho^1 \otimes \varrho^2 \otimes \varrho^3\right), \tag{4.1}$$

where the maximization runs over all normalized complete product states. Theorem 1 of Ref.[64] states that the maximization of a pure state over a single qubit state can be completely derived by using a particle traced over density matrix. Hence the theorem allows us to re-express the entanglement eigenvalue by reduced density matrix ρ^{AB} of qubits A and B

$$\Lambda^2_{\max} = \max_{\varrho^1 \varrho^2} \operatorname{tr}\left(\rho^{AB} \varrho^1 \otimes \varrho^2\right). \tag{4.2}$$

Now we introduce four Bloch vectors:

1) r_A for the reduced density matrix ρ^A of the qubit A,

2) r_B for the reduced density matrix ρ^B of the qubit B,

3) u for the single qubit state ϱ^1,

4) s for the single qubit state ϱ^2.

Then the expression for entanglement eigenvalue (4.2) takes the form

$$\Lambda^2_{\max} = \frac{1}{4} \max_{u^2=v^2=1} \left(1 + u \cdot r_A + s \cdot r_B + g_{ij}\, u_i v_j\right), \tag{4.3}$$

where(summation on repeated indices i and j is understood)

$$g_{ij} = \operatorname{tr}(\rho^{AB} \sigma_i \otimes \sigma_j) \tag{4.4}$$

and σ_i's are Pauli matrices. The closest product state satisfies the stationarity conditions

$$r_A + gs = \lambda_1 u, \quad r_B + g^T u = \lambda_2 s, \tag{4.5}$$

where Lagrange multipliers λ_1 and λ_2 enforce the unit Bloch vectors u and s. The solutions of Eq.(4.5) are

$$u = \left(\lambda_1 \lambda_2 \mathbb{1} - g\, g^T\right)^{-1} \left(\lambda_2 r_A + g\, r_B\right), \quad s = \left(\lambda_1 \lambda_2 \mathbb{1} - g^T g\right)^{-1} \left(\lambda_1 r_B + g^T r_A\right). \tag{4.6}$$

Unknown Lagrange multipliers are defined by equations

$$u^2 = 1, \quad v^2 = 1. \tag{4.7}$$

In general, Eq.(4.7) gives algebraic equations of degree six. The reason for this is that stationarity equations define all extremes of the reduced density matrix ρ^{AB} over product states, regardless of them being global or local. And the degree of the algebraic equations is the number of possible extremes.

Eq.(4.6) contains valuable information. It provides solid bases for a new numerical approach. This can be compared with the numerical calculations based on other technique [93].

4.2 Three Qubit State

We consider W-type state

$$|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle + d|111\rangle, \tag{4.8}$$

where free parameters a, b, c, d satisfy the normalization condition $a^2+b^2+c^2+d^2 = 1$. Without loss of generality we consider only the case of positive parameters a, b, c, d. At first sight, it is not obvious whether the state allows analytic solutions or not. However, it does and our first task is to confirm the existence of the analytic solutions.

In fact, entanglement of the state Eq.(4.8) is invariant under the permutations of four parameters a, b, c, d. The invariance under the permutations of three parameters a, b, c is the consequence of the invariance under the permutations of qubits A,B,C. Now we make a local unitary(LU) transformation that relabels the bases of qubits B and C, i.e. $0_B \leftrightarrow 1_B,\ 0_C \leftrightarrow 1_C$, and does not change the basis of qubit A. This LU-transformation interchanges the coefficients as follows: $a \leftrightarrow d,\ b \leftrightarrow c$. Since any entanglement measure must be invariant under LU-transformations and the permutation $b \leftrightarrow c$, it must be also invariant under the permutation $a \leftrightarrow d$. In view of this symmetry, any entanglement measure must be invariant under the permutations of all the state parameters a, b, c, d. Owing to this symmetry, the state allows to derive analytic expressions for the entanglement eigenvalues. The necessary condition is [66]

$$\det\left(\lambda_1\lambda_2 \mathbb{1} - g\, g^T\right) = 0. \tag{4.9}$$

Indeed, if the condition (4.9) is fulfilled, then the expressions (4.6) for the general solutions are not applicable and Eq.(4.5) admits further simplification.

Denote by i, j, k unit vectors along axes x, y, z respectively. Straightforward calculation yields

$$r_A = r_1\, k, \quad r_B = r_2\, k, \quad g = \begin{pmatrix} 2\omega & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & -r_3 \end{pmatrix}, \tag{4.10}$$

where

$$\begin{aligned} &r_1 = b^2 + c^2 - a^2 - d^2, \quad r_2 = a^2 + c^2 - b^2 - d^2, \\ &r_3 = a^2 + b^2 - c^2 - d^2, \quad \omega = ab + dc, \quad \mu = ab - dc. \end{aligned} \tag{4.11}$$

Vectors u and s can be written as linear combinations

$$u = u_i i + u_j j + u_k k, \quad s = v_i i + v_j j + v_k k \tag{4.12}$$

of vectors i, j, k. The substitution of the Eq.(4.12) into Eq.(4.5) gives a couple of equations in each direction. The result is a system of six linear equations

$$2\omega\, v_i = \lambda_1 u_i, \quad 2\omega\, u_i = \lambda_2 v_i, \tag{4.13a}$$

$$2\mu\, v_j = \lambda_1 u_j, \quad 2\mu\, u_j = \lambda_2 v_j, \tag{4.13b}$$

$$r_1 - r_3 v_k = \lambda_1 u_k, \quad r_2 - r_3 u_k = \lambda_2 v_k. \tag{4.13c}$$

Above equations impose two conditions

$$(\lambda_1\lambda_2 - 4\omega^2) u_i v_i = 0, \tag{4.14a}$$

$$(\lambda_1\lambda_2 - 4\mu^2) u_j v_j = 0. \tag{4.14b}$$

From these equations it can be deduced that the condition (4.9) is valid and the system of equations (4.5) and (4.7) is solvable. Note that as a consequences of Eq.(4.13) x and/or y components of vectors u and s vanish simultaneously. Hence, conditions (4.14) are satisfied in following three cases:

- vectors u and s lie in xz plane

$$\lambda_1\lambda_2 - 4\omega^2 = 0, \quad u_j v_j = 0, \tag{4.15}$$

- vectors u and s lie in yz plane

$$\lambda_1\lambda_2 - 4\mu^2 = 0, \quad u_i v_i = 0, \tag{4.16}$$

- vectors u and s are aligned with axis z

$$u_i v_i = u_j v_j = 0. \tag{4.17}$$

These cases are examined individually in next section.

4.3 Explicit expressions

In this section we analyze all three cases and derive explicit expressions for entanglement eigenvalue. Each expression has its own range of definition in which they are deemed applicable. Three ranges of definition cover the four dimensional sphere given by normalization condition. It is necessary to separate the validity domains and to make clear which of expressions should be applied for a given state. It turns out that the separation of domains requires solving inequalities that contain polynomials of degree six. This is a nontrivial task and we investigate it in the next section.

4.3.1 Circumradius of Convex Quadrangle

Let us consider the first case. Our main task is to find Lagrange multipliers λ_1 and λ_2. From equations (4.13c) and (4.15) we have

$$u_k = \frac{\lambda_2 r_1 - r_2 r_3}{4\omega^2 - r_3^2}, \quad v_k = \frac{\lambda_1 r_2 - r_1 r_3}{4\omega^2 - r_3^2}. \tag{4.18}$$

In its turn Eq.(4.13a) gives

$$\lambda_1 u_i^2 = \lambda_2 v_i^2. \tag{4.19}$$

Eq.(4.7) allows the substitution of expressions (4.18) into Eq.(4.19). Then we can obtain the second equation for Lagrange multipliers

$$\lambda_1 \left(4\omega^2 + r_2^2 - r_3^2\right) = \lambda_2 \left(4\omega^2 + r_1^2 - r_3^2\right). \tag{4.20}$$

This equation has a simple form owing to condition (4.9). Thus we can factorize the equation of degree six into the quadratic equations. Equations (4.20) and (4.15) together yield

$$\lambda_1 = 2\omega\,\frac{bc + ad}{ac + bd}, \quad \lambda_2 = 2\omega\,\frac{ac + bd}{bc + ad}. \tag{4.21}$$

Note that we kept only positive values of Lagrange multipliers and omitted negative values to get the maximal value of $\Lambda^2_{\max}$. Now Eq.(4.3) takes the form

$$4\Lambda^2_{\max} = 1 + \frac{8(ab + cd)(ac + bd)(ad + bc) - r_1 r_2 r_3}{4\omega^2 - r_3^2}. \tag{4.22}$$

In fact, entanglement eigenvalue is the sum of two equal terms and this statement follows from the identity

$$1 - \frac{r_1 r_2 r_3}{4\omega^2 - r_3^2} = 8\frac{(ab+cd)(ac+bd)(ad+bc)}{4\omega^2 - r_3^2}. \tag{4.23}$$

To derive this identity one has to use the normalization condition $a^2+b^2+c^2+d^2 = 1$. The identity allows to rewrite Eq.(4.22) as follows

$$\Lambda_{\max}^2 = 4R_q^2, \tag{4.24}$$

where

$$R_q^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{4\omega^2 - r_3^2}. \tag{4.25}$$

Above formula has a geometric interpretation and now we demonstrate it. Let us define a quantity $p \equiv (a+b+c+d)/2$. Then the denominator can be rewritten as

$$4\omega^2 - r_3^2 = 16(p-a)(p-b)(p-c)(p-d). \tag{4.26}$$

Five independent parameters are necessary to construct a convex quadrangle. However, four independent parameters are necessary to construct a convex quadrangle that has circumradius. For such quadrangles the area S_q is given exactly by Eq.(4.26) up to numerical factor, that is $S_q^2 = (p-a)(p-b)(p-c)(p-d)$. Hence Eq.(4.25) can be rewritten as

$$R_q^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{16S_q^2}. \tag{4.27}$$

Thus R_q can be interpreted as a circumradius of the convex quadrangle. Eq.(4.27) is the generalization of the corresponding formula of Ref.[66] and reduces to the circumradius of the triangle if one of parameters is zero.

Eq.(4.24) is valid if vectors u and s are unit and have non-vanishing x components. These conditions have short formulations

$$|u_k| \leq 1, \quad |v_k| \leq 1. \tag{4.28}$$

Above inequalities are polynomials of degree six and algebraic solutions are unlikely. However, it is still possible do define the domain of validity of Eq.(4.27).

4.3.2 Circumradius of Crossed-Quadrangle

Here, we consider the second case given by Eq.(4.16). Derivations repeat steps of the previous subsection and the only difference is the interchange $\omega \leftrightarrow \mu$. Therefore we skip some obvious steps and present only main results. Components of vectors u and s along axis z are

$$u_k = \frac{\lambda_2 r_1 - r_2 r_3}{4\mu^2 - r_3^2}, \quad v_k = \frac{\lambda_1 r_2 - r_1 r_3}{4\mu^2 - r_3^2}. \tag{4.29}$$

The second equation for Lagrange multipliers

$$\lambda_1 \left(4\mu^2 + r_2^2 - r_3^2\right) = \lambda_2 \left(4\mu^2 + r_1^2 - r_3^2\right) \tag{4.30}$$

together with Eq.(4.16) yields

$$\lambda_1 = \pm 2\mu \, \frac{bc - ad}{ac - bd}, \quad \lambda_2 = \pm 2\mu \, \frac{ac - bd}{bc - ad}. \tag{4.31}$$

Using these expressions, one can derive the following expression for entanglement eigenvalue

$$4\Lambda^2_{\max} = 1 + \frac{\lambda_2(4\mu^2 + r_1^2 - r_3^2) - r_1 r_2 r_3}{4\mu^2 - r_3^2}. \tag{4.32}$$

Now the restrictions $1/4 < \Lambda^2_{\max} \leq 1$ derived in Ref.[64] uniquely define the signs in Eq.(4.31). Right signs enforce strictly positive fraction in right hand side of Eq.(4.32). To make a right choice, we replace d by $-d$ in the identity (4.23) and rewrite Eq.(4.32) as follows

$$4\Lambda^2_{\max} = \frac{1}{2} \, \frac{(ac - bd)(bc - ad)(ab - cd)}{p(p - c - d)(p - b - d)(p - a - d)} \pm \frac{1}{2} \, \frac{(ac - bd)(bc - ad)(ab - cd)}{p(p - c - d)(p - b - d)(p - a - d)}. \tag{4.33}$$

Lower sign yields zero and is wrong. It shows that reduced density matrix ρ^{AB} still has zero eigenvalue.

Upper sign may yield a true answer. Entanglement eigenvalue is

$$\Lambda^2_{\max} = 4R^2_{\times}, \tag{4.34}$$

where

$$R^2_{\times} = \frac{(ac - bd)(bc - ad)(ab - cd)}{16 S^2_{\times}}, \tag{4.35}$$

and $S^2_{\times} = p(p - c - d)(p - b - d)(p - a - d)$. The formula (4.35) may seem suspicious because it is not clear whether right hand side is positive and lies in required region. To clarify the situation we present a geometrical treatment of Eq.(4.35).

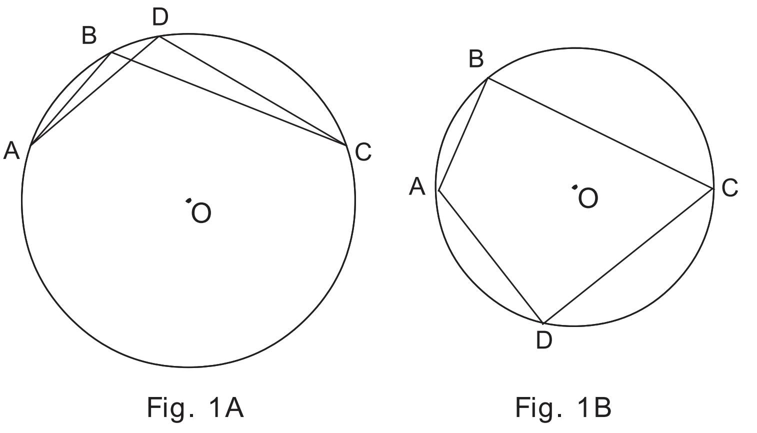

Figure 4.1: This figure shows the example for the case when crossed quadrangle(Fig.1A) has larger circumradius than that of convex quadrangle(Fig.1B) with same sides.

The geometrical figure $ABCD$ in Fig.1A is not a quadrangle and is not a polygon at all. The reason is that it has crossed sides AD and BC. We call figure $ABCD$ crossed-quadrangle in a figurative sense as it has four sides and a cross point. Another justification of this term is that we will compare figure $ABCD$ in Fig.1A with a convex quadrangle $ABCD$ containing the same sides.

Consider a crossed-quadrangle $ABCD$ with sides $AB = a, BC = b, CD = c, DA = d$ that has circumcircle. It is easy to find the length of the interval AC

$$AC^2 = \frac{(ac - bd)(bc - ad)}{ab - cd}. \tag{4.36}$$

This relation is true unless triangles ABC and ADC have the same height and as a consequence equal areas. Note that $S_\times$ is not an area of the crossed-quadrangle. It is the difference between the areas of the noted triangles.

Using Eq.(4.36), one can derive exactly Eq.(4.35) for the circumradius of the crossed-quadrangle.

Eq.(4.34) is meaningful if vectors u and s are unit and have nonzero components along the axis y.

4.3.3 Largest Coefficient

In this subsection we consider the last case described by Eq.(4.17). Entanglement eigenvalue takes maximal value if all terms in r.h.s. of Eq.(4.3) are positive. Then equations (4.17) and (4.10) together impose

$$u = \mathrm{Sign}(r_1)k, \quad s = \mathrm{Sign}(r_2)k, \quad r_1 r_2 r_3 < 0, \tag{4.37}$$

where Sign(x) gives -1, 0 or 1 depending on whether x is negative, zero, or positive. Substituting these values into Eq.(4.3), we obtain

$$\Lambda^2_{\max} = \frac{1}{4}\left(1 + |r_1| + |r_2| + |r_3|\right). \tag{4.38}$$

Owing to inequality, $r_1 r_2 r_3 < 0$, above expression always gives a square of the largest coefficient l

$$l = \max(a, b, c, d) \tag{4.39}$$

in Eq.(4.8). Indeed, let us consider the case $r_1 > 0, r_2 > 0, r_3 < 0$. From inequalities $r_1 > 0, r_2 > 0$ it follows that $c^2 > d^2 + |a^2 - b^2|$ and therefore $c^2 > d^2$. Note, $c^2 > d^2$ is necessary but not sufficient condition. Now if $d > b$, then $r_1 > 0$ yields $c > a$ and if $d < b$, then $r_3 < 0$ yields $c > a$. Thus inequality $c > a$ is true in all cases. Similarly $c > b$ and c is the largest coefficient. On the other hand $\Lambda^2_{\max} = c^2$ and Eq.(4.38) really gives the largest coefficient in this case.

Similarly, cases $r_1 > 0, r_2 < 0, r_3 > 0$ and $r_1 < 0, r_2 > 0, r_3 > 0$ yield $\Lambda^2_{\max} = b^2$ and $\Lambda^2_{\max} = a^2$, respectively. And again entanglement eigenvalue takes the value of the largest coefficient.

The last possibility $r_1 < 0, r_2 < 0, r_3 < 0$ can be analyzed using analogous speculations. One obtains $\Lambda^2_{\max} = d^2$ and d is the largest coefficient.

Combining all cases mentioned earlier, we rewrite Eq.(4.38) as follows

$$\Lambda^2_{\max} = l^2. \tag{4.40}$$

This expression is valid if both vectors u and s are collinear with the axes z.

We have derived three expressions for (4.24),(4.34) and (4.40) for entanglement eigenvalue. They are valid when vectors u and s lie in xz plane, lie in yz plane and are collinear with axis z, respectively. The following section goes on to specify these domains by parameters a, b, c, d.

4.4 Applicable Domains

Mainly, two points are being analyzed. First, we probe into the meaningful geometrical interpretations of quantities R_q and $R_\times$. Second, we separate validity domains of equations (4.24),(4.34) and (4.40). It is mentioned earlier that algebraic methods for solving the inequalities of degree six are ineffective. Hence, we use geometric tools that are elegant and concise in this case.

We consider four parameters a, b, c, d as free parameters as the normalization condition is irrelevant here. Indeed, one can use the state $|\psi\rangle/\sqrt{a^2+b^2+c^2+d^2}$ where all parameters are free. If one repeats the same steps, the only difference is that the entanglement eigenvalue $\Lambda^2_{\max}$ is replaced by $\Lambda^2_{\max}/(a^2+b^2+c^2+d^2)$. In other words, normalization condition re-scales the quadrangle, convex or crossed, so that the circumradius always lies in the required region. Consequently, in constructing quadrangles we can neglect the normalization condition and consider four free parameters a, b, c, d.

4.4.1 Existence of circumcircle.

It is known that four sides a, b, c, d of the convex quadrangle must obey the inequality $p - l > 0$. Any set of such parameters forms a cyclic quadrilateral. Note that the quadrangle is not unique as the sides can be arranged in different orders. But all these quadrangles have the same circumcircle and the circumradius is unique.

The sides of a crossed-quadrangle must obey the same condition. Indeed, from Fig.1A it follows that $BC-AB < AC < AD+DC$ and $DC-AD < AC < AB+BC$. Therefore $AB + AD + DC > BC$ and $AB + BC + AD > DC$. The sides BC and DC are two largest sides and consequently $p - l > 0$. However, the existence of the circumcircle requires an additional condition and it is explained here. The relation $r_3 = 2\mu \cos ABC$ forces $4\mu^2 \geq r_3^2$ and, therefore

$$S^2_\times \geq 0. \tag{4.41}$$

Thus the denominator in Eq.(4.35) must be positive. On the other hand the inequality $AC^2 \geq 0$ forces a positive numerator of the same fraction

$$(ac - bd)(bc - ad)(ab - cd) \geq 0. \tag{4.42}$$

These two inequalities impose conditions on parameters a, b, c, d. For the future considerations, we need to write explicitly the condition imposed by inequality (4.42). The numerator is a symmetric function on parameters a, b, c, d and it suffices to analyze only the case $a \geq b \geq c \geq d$. Obviously $(ac - bd) \geq 0$, $(ab - cd) \geq 0$ and it remains the constraint $bc \geq ad$. The last inequality states that the product of the largest and smallest coefficients must not exceed the product of remaining coefficients. Denote by s the smallest coefficient

$$s = \min(a, b, c, d). \tag{4.43}$$

We can summarize all cases as follows

$$l^2 s^2 \leq abcd. \tag{4.44}$$

This is necessary but not sufficient condition for the existence of $R_\times$. The next condition $S_\times^2 > 0$ we do not analyze because the first condition (4.44) suffices to separate the validity domains.

4.4.2 Separation of validity domains.

In this section we define applicable domains of expressions (4.24),(4.34) and (4.40) step by step.

Circumradius of convex quadrangle. First we separate the validity domains between the convex quadrangle and the largest coefficient. In a highly entangled region, where the center of circumcircle lies inside the quadrangle, the circumradius is greater than any of sides and yield a correct answer. This situation is changed when the center lies on the largest side of the quadrangle and both equations (4.24) and (4.40) give equal answers. Suppose that the side a is the largest one and the center lies on the side a. A little geometrical speculation yields

$$a^2 = b^2 + c^2 + d^2 + 2\frac{bcd}{a}. \tag{4.45}$$

From this equation we deduce that if a^2 is smaller than r.h.s., *i.e.*

$$a^2 \leq b^2 + c^2 + d^2 + 2\frac{bcd}{a}, \tag{4.46}$$

then the circumradius-formula is valid. If a^2 is greater than r.h.s in Eq.(4.45), then the largest coefficient formula is valid. The inequality (4.46) also guarantees the existence of the cyclic quadrilateral. Indeed, using the inequality

$$bc + cd + bd \geq 3\frac{bcd}{a}, \tag{4.47}$$

one derives

$$(b+c+d)^2 \geq b^2+c^2+d^2+\frac{6bcd}{a} \geq a^2. \tag{4.48}$$

Above inequality ensures the existence of a convex quadrangle with the given sides.

To get a confidence, we can solve equation $u_k = \pm 1$ using the relation (4.45). However, it is more transparent to factorize it as following:

$$(4\omega^2 - r_3^2)(1+u_k) = \frac{2ad}{bc+ad}\left(b^2+c^2+d^2+\frac{2bcd}{a}-a^2\right)\left(a^2+b^2+c^2+\frac{2abc}{d}-d^2\right) \tag{4.49a}$$

$$(4\omega^2 - r_3^2)(1-u_k) = \frac{2bc}{bc+ad}\left(a^2+c^2+d^2+\frac{2acd}{b}-b^2\right)\left(a^2+b^2+d^2+\frac{2abd}{c}-c^2\right). \tag{4.49b}$$

Similarly, we have

$$(4\omega^2 - r_3^2)(1+v_k) = \frac{2bd}{ac+bd}\left(a^2+c^2+d^2+\frac{2acd}{b}-b^2\right)\left(a^2+b^2+c^2+\frac{2abc}{d}-d^2\right) \tag{4.50a}$$

$$(4\omega^2 - r_3^2)(1-v_k) = \frac{2ac}{ac+bd}\left(b^2+c^2+d^2+\frac{2bcd}{a}-a^2\right)\left(a^2+b^2+d^2+\frac{2abd}{c}-c^2\right). \tag{4.50b}$$

Thus, the circumradius of the convex quadrangle gives a correct answer if all brackets in the above equations are positive. In general, Eq.(4.24) is valid if

$$l^2 \leq \frac{1}{2}+\frac{abcd}{l^2}. \tag{4.51}$$

When one of parameters vanishes, i.e. $abcd = 0$, inequality (4.51) coincides with the corresponding condition in Ref.[66].

Circumradius of crossed quadrangle. Next we separate the validity domains between the convex and the crossed quadrangles. If $S_\times^2 < 0$, then crossed one has no circumcircle and the only choice is the circumradius of the convex quadrangle. If $S_\times^2 > 0$, then we use the equality

$$4R_q^2 - 4R_\times^2 = \frac{r}{2}\frac{abcd}{S_q^2 S_\times^2} \tag{4.52}$$

where $r = r_1 r_2 r_3$. It shows that $r > 0$ yields $R_q > R_\times$ and vice-versa. Entanglement eigenvalue always takes the maximal value. Therefore, $\Lambda^2_{\max} = 4R_q^2$ if $r > 0$ and

$\Lambda^2_{\max} = 4R^2_\times$ if $r < 0$. Thus $r = 0$ is the separating surface and it is necessary to analyze the condition $r < 0$.

Suppose $a \geq b \geq c \geq d$. Then r_2 and r_3 are positive. Therefore r is negative if and only if r_1 is negative, which implies

$$a^2 + d^2 > b^2 + c^2. \tag{4.53}$$

Now suppose $a \geq d \geq b \geq c$. Then r_1 is negative and r_3 is positive. Therefore r_2 must be positive, which implies

$$a^2 + c^2 > b^2 + d^2. \tag{4.54}$$

It is easy to see that in both cases left hand sides contain the largest and smallest coefficients. This result can be generalized as follows: $r \leq 0$ if and only if

$$l^2 \geq \frac{1}{2} - s^2. \tag{4.55}$$

It remains to separate the validity domains between the crossed-quadrangle and the largest coefficient. We can use three equivalent ways to make this separation:

1)to use the geometric picture and to see when $4R^2_\times$ and l^2 coincide,

2)directly factorize equation $u_k = \pm 1$,

3)change the sign of the parameter d.

All of these give the same result stating that Eq.(4.34) is valid if

$$l^2 \leq \frac{1}{2} - \frac{abcd}{l^2}. \tag{4.56}$$

Inequalities (4.55) and (4.56) together yield

$$l^2 s^2 \geq abcd. \tag{4.57}$$

This inequality is contradicted by (4.44) unless $l^2 s^2 = abcd$. Special cases like $l^2 s^2 = abcd$ are considered in the next section. Now we would like to comment the fact that crossed quadrangle survives only in exceptional cases. Actually crossed case can be obtained from the convex cases by changing the sign of any parameter. It crucially depends on signs of parameters or, in general, on phases of parameters. On the other hand all phases in Eq.(4.8) can be eliminated by LU-transformations. For example, the phase of d can be eliminated by redefinition of the phase of the state function $|\psi\rangle$ and the phases of remaining parameters can be absorbed in the definitions of basis vectors $|1\rangle$ of the qubits A, B and C. Owing to this entanglement eigenvalue being

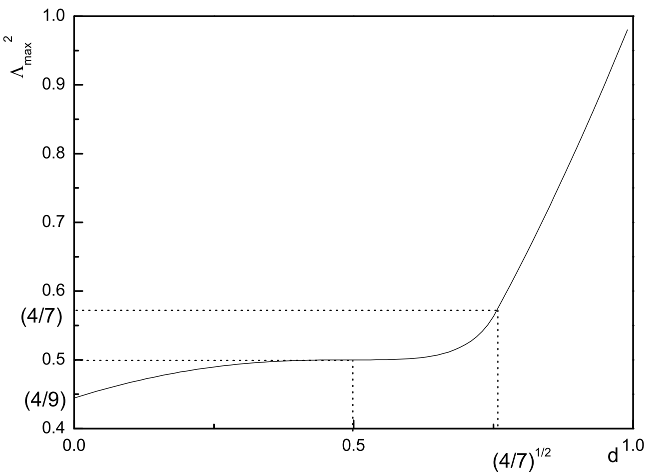

Figure 4.2: Plot of d-dependence of Λ^2_{max} when $a = b = c$. When $d \to 1$, $\Lambda^2_{\max}$ goes to 1 as expected. When $d = 0$, $\Lambda^2_{\max}$ becomes 4/9, which coincides with the result of Ref.[43]. When $r = 0$ which implies $a = d = 1/2$, $\Lambda^2_{\max}$ becomes 1/2 (it is shown as dotted line). When $d = 2a$, which implies $d = \sqrt{4/7}$, $\Lambda^2_{\max}$ goes to 4/7, which is one of shared states (it is also shown as another dotted line).

LU invariant quantity does not depend on phases. However, crossed case is relevant if one considers states given by Generalized Schmidt Decomposition(GSD) [68]. In this case phases can not be gauged away and crossed case has its own range of definition. This range has shrunk to the separating surface $r = 0$ in our case.

Now we are ready to present a distinct separation of the validity domains:

$$\Lambda^2_{\max} = \begin{cases} 4R_q^2, & \text{if} \quad l^2 \le 1/2 + abcd/l^2 \\ l^2 & \text{if} \quad l^2 \ge 1/2 + abcd/l^2 \end{cases} \tag{4.58}$$

As an illustration we present the plot of d-dependence of $\Lambda^2_{\max}$ in Fig.2 when $a = b = c$.

We have distinguished three types of quantum states depending on which expression takes entanglement eigenvalue. Also there are states that lie on surfaces separating different applicable domains. They are shared by two types of quantum states and may have interesting features. We will call those shared states. Such shared states are considered in the next section.

4.5 Shared States.

Consider quantum states for which both convex and crossed quadrangles yield the same entanglement eigenvalue. Eq.(4.36) is not applicable and we rewrite equations (4.27) and (4.35) as follows

$$4R_q^2 = \frac{1}{2}\left(1 - \frac{r}{16S_q^2}\right), \quad 4R_\times^2 = \frac{1}{2}\left(1 - \frac{r}{16S_\times^2}\right). \tag{4.59}$$

These equations show that if the state lies on the separating surface $r = 0$, then entanglement eigenvalue is a constant

$$\Lambda_{\max}^2 = \frac{1}{2} \tag{4.60}$$

and does not depend on the state parameters. This fact has a simple interpretation. Consider the case $r_1 = 0$. Then $b^2 + c^2 = a^2 + d^2 = 1/2$ and the quadrangle consists of two right triangles. These two triangles have a common hypotenuse and legs b, c and a, d, respectively, regardless of the triangles being in the same semicircle or in opposite semicircles. In both cases they yield same circumradius. Decisive factor is that the center of the circumcircle lies on the diagonal. Thus the perimeter and diagonals of the quadrangle divide ranges of definition of the convex quadrangle. When the center of circumcircle passes the perimeter, entanglement eigenvalue changes-over from convex circumradius to the largest coefficient. And if the center lies on the diagonal, convex and crossed circumradiuses become equal.

We would like to bring plausible arguments that this picture is incomplete and there is a region that has been shrunk to the point. Consider three-qubit state given by GSD

$$|\psi\rangle = a|100\rangle + b|010\rangle + b|001\rangle + d|111\rangle + e|000\rangle. \tag{4.61}$$

One of parameters must have non-vanishing phase[68] and we can treat this phase as an angle. Then, we have five sides and an angle. This set defines a sexangle that has circumcircle. One can guess that in a highly entangled region entanglement eigenvalue is the circumradius of the sexangle. However, there is a crucial difference. Any convex sexangle contains a star type area and the sides of this area are the diagonals of the sexangle. The perimeter of the star separates the convex and the crossed cases.

Unfortunately, we can not see this picture in our case because the diagonals of a quadrangle confine a single point. It is left for future to calculate the entanglement eigenvalues for arbitrary three qubit states and justify this general picture.

Shared states given by $r = 0$ acquire new properties. They can be used for perfect teleportation and superdense coding [66, 105, 115]. This statement is not proven clearly, but also no exceptions are known.

Now consider a case where the largest coefficient and circumradius of the convex quadrangle coincide with each other. The separating surface is given by

$$l^2 = \frac{1}{2} + \frac{abcd}{l^2}. \tag{4.62}$$

Entanglement eigenvalue ranges within the narrow interval

$$\frac{1}{2} \leq \Lambda^2_{\max} \leq \frac{4}{7}. \tag{4.63}$$

It separates slightly and highly entangled states. When one of coefficients is large enough and satisfies the relation $l^2 > 1/2 + abcd/l^2$, entanglement eigenvalue takes a larger coefficient. And the expression (4.8) for the state function effectively takes the place of Schmidt decomposition. In highly entangled region no similar picture exists and all coefficients participate in equal parts and yield the circumradius. Thus, shared states given by Eq.(4.62) separate slightly entangled states from highly entangled ones, and can be ascribed to both types.

What is the meaning of these states? Shared states given by $r = 0$ acquire new and important features. One can expect that shared states dividing highly and slightly entangled states also must acquire some new features. However, these features are yet to be discovered.

4.6 Conclusions

We have considered three-parametric families of three qubit states and derived explicit expressions for entanglement eigenvalue. The final expressions have their own geometrical interpretation. The result in this chapter with the results of Ref.[66] show that the geometric measure has two visiting cards: the circumradius and the largest coefficient. The geometric interpretation may enable us to predict the answer for the states given by GSD. If the center of circumcircle lies in star type area confined by diagonals of the sexangle, then entanglement eigenvalue is the circumradius of the crossed sexangle(s). If the center lies in the remaining part of sexangle, the entanglement eigenvalue is the circumradius of the convex sexangle. And when the center passes the perimeter, then entanglement eigenvalue is the largest coefficient. Although we cannot justify our prediction due to lack of computational technique, this picture surely enables us to take a step toward a deeper understanding of the entanglement measure [69].

Shared states given by $r = 0$ play an important role in quantum information theory. The application of shared states given by Eq.(4.62) is somewhat questionable, and should be analyzed further. It should be pointed out that one has to understand the properties of these states and find the possible applications. We would like to investigate this issue elsewhere.

Finally following our procedure, one can obtain the nearest product state of a given three-parametric W-type state. These two states will always be separated by a line of densities composed of the convex combination of W-type states and the nearest product states [116]. There is a separable density matrix ϱ_0 which splits the line into two parts as follows. One part consists of separable densities and another part consists of non-separable densities. It was shown in Ref.[116] that an operator $W = \varrho_0 - \rho^{ABC} - \mathrm{tr}[\varrho_0(\varrho_0 - \rho^{ABC})]I$ has the properties $\mathrm{tr}(W\rho^{ABC}) < 0$, and $\mathrm{tr}(W\varrho) \geq 0$ for the arbitrary separable state ϱ. The operator W is clearly Hermitian and thus is an entanglement witness for the state. Thus our results allow oneself to construct the entanglement witnesses for W-type three qubit states. However, the explicit derivation of ϱ_0 seems to be highly non-trivial [117, 118].

Chapter 5

Three-qubit Groverian measure

Recently, much attention is paid to quantum entanglement[119]. It is believed in quantum information community that entanglement is the physical resource which makes quantum computer outperforms classical one[32]. Thus in order to exploit fully this physical resource for constructing and developing quantum algorithms it is important to quantify the entanglement. The quantity for the quantification is usually called entanglement measure.

About decade ago the axioms which entanglement measures should satisfy were studied[33]. The most important property for measure is monotonicity under local operation and classical communication(LOCC)[34]. Following the axioms, many entanglement measures were constructed such as relative entropy[35], entanglement of distillation[36] and formation[37, 38, 39, 40], geometric measure[41, 42, 43, 116], Schmidt measure[108] and Groverian measure[46]. Entanglement measures are used in various branches of quantum mechanics. Especially, recently, they are used to try to understand Zamolodchikov's c-theorem[47] more profoundly. It may be an important application of the quantum information techniques to understand the effect of renormalization group in field theories[48].

The purpose of this chapter is to compute the Groverian measure for various three-qubit quantum states.The Groverian measure $G(\psi)$ for three-qubit state $|\psi\rangle$ is defined by $G(\psi) \equiv \sqrt{1 - P_{max}}$ where

$$P_{max} = \max_{|q_1\rangle, |q_2\rangle, |q_3\rangle} |\langle q_1|\langle q_2|\langle q_3|\psi\rangle|^2. \tag{5.1}$$

Thus P_{max} can be interpreted as a maximal overlap between the given state $|\psi\rangle$ and product states. Groverian measure is an operational treatment of a geometric measure. Thus, if one can compute $G(\psi)$, one can also compute the geometric measure of pure state by $G^2(\psi)$. Sometimes it is more convenient to re-express Eq.(5.1) in terms of the density matrix $\rho = |\psi\rangle\langle\psi|$. This can be easily accomplished by an expression

$$P_{max} = \max_{R^1, R^2, R^3} \mathrm{Tr}\left[\rho R^1 \otimes R^2 \otimes R^3\right] \tag{5.2}$$

where $R^i \equiv |q_i\rangle\langle q_i|$ density matrix for the product state. Eq.(5.1) and Eq.(5.2) manifestly show that P_{max} and $G(\psi)$ are local-unitary(LU) invariant quantities. Since it is well-known that three-qubit system has five independent LU-invariants[71, 68, 120], *say* $J_i (i = 1, \cdots, 5)$, we would like to focus on the relation of the Groverian measures to LU-invariants J_i's in this chapter.

This chapter is organized as follows. In section 5.1 we review simple case, *i.e.* two-qubit system. Using Bloch form of the density matrix it is shown in this section that two-qubit system has only one independent LU-invariant quantity, *say* J. It is also shown that Groverian measure and P_{max} for arbitrary two-qubit states can be expressed solely in terms of J. In section 5.2 we have discussed how to derive LU-invariants in higher-qubit systems. In fact, we have derived many LU-invariant quantities using Bloch form of the density matrix in three-qubit system. It is shown that all LU-invariants derived can be expressed in terms of J_i's discussed in Ref.[68]. Recently, it was shown in Ref.[64] that P_{max} for n-qubit state can be computed from $(n-1)$-qubit reduced mixed state. This theorem was used in Ref.[66] and Ref.[67] to compute analytically the geometric measures for various three-qubit states. In this section we have discussed the physical reason why this theorem is possible from the aspect of LU-invariance. In section 5.3 we have computed the Groverian measures for various types of the three-qubit system. The five types we discussed in this section were originally developed in Ref.[68] for the classification of the three-qubit states. It has been shown that the Groverian measures for type 1, type 2, and type 3 can be analytically computed. We have expressed all analytical results in terms of LU-invariants J_i's. For type 4 and type 5 the analytical computation seems to be highly nontrivial and may need separate publications. Thus the analytical calculation for these types is not presented in this chapter. The results of this section are summarized in Table I. In section 5.4 we have discussed the modified W-like state, which has three-independent real parameters. In fact, this state cannot be categorized in the five types discussed in section 5.3. The analytic expressions of the Groverian measure for this state was computed recently in Ref.[67]. It was shown that the measure has three different expressions depending on the domains of the parameter space. It turned out that each expression has its own geometrical meaning. In this section we have re-expressed all expressions of the Groverian measure in terms of LU-invariants. In section 5.5 brief conclusion is given.

5.1 Two Qubit: Simple Case

In this section we consider P_{max} for the two-qubit system. The Groverian measure for two-qubit system is already well-known[93]. However, we revisit this issue here to explore how the measure is expressed in terms of the LU-invariant quantities. The Schmidt decomposition[121, 122] makes the most general expression of the two-qubit state vector to be simple form

$$|\psi\rangle = \lambda_0|00\rangle + \lambda_1|11\rangle \tag{5.3}$$

with $\lambda_0, \lambda_1 \geq 0$ and $\lambda_0^2 + \lambda_1^2 = 1$. The density matrix for $|\psi\rangle$ can be expressed in the Bloch form as following:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{4}\left[\mathbb{1}\otimes\mathbb{1} + v_{1\alpha}\sigma_\alpha\otimes\mathbb{1} + v_{2\alpha}\mathbb{1}\otimes\sigma_\alpha + g_{\alpha\beta}\sigma_\alpha\otimes\sigma_\beta\right], \tag{5.4}$$

where

$$\vec{v}_1 = \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ \lambda_0^2 - \lambda_1^2 \end{pmatrix}, \qquad g_{\alpha\beta} = \begin{pmatrix} 2\lambda_0\lambda_1 & 0 & 0 \\ 0 & -2\lambda_0\lambda_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.5}$$

In order to discuss the LU transformation we consider first the quantity $U\sigma_\alpha U^\dagger$ where U is 2×2 unitary matrix. With direct calculation one can prove easily

$$U\sigma_\alpha U^\dagger = \mathcal{O}_{\alpha\beta}\sigma_\beta, \tag{5.6}$$

where the explicit expression of $\mathcal{O}_{\alpha\beta}$ is given in appendix A. Since $\mathcal{O}_{\alpha\beta}$ is a real matrix satisfying $\mathcal{O}\mathcal{O}^T = \mathcal{O}^T\mathcal{O} = \mathbb{1}$, it is an element of the rotation group O(3). Therefore, Eq.(5.6) implies that the LU-invariants in the density matrix (5.4) are $|\vec{v}_1|$, $|\vec{v}_2|$, $\mathrm{Tr}[gg^T]$ etc.

All LU-invariant quantities can be written in terms of one quantity, *say* $J \equiv \lambda_0^2\lambda_1^2$. In fact, J can be expressed in terms of two-qubit concurrence[39] $\mathcal{C}$ by $\mathcal{C}^2/4$. Then it is easy to show

$$\begin{aligned} |\vec{v}_1|^2 = |\vec{v}_2|^2 = 1 - 4J, \\ g_{\alpha\beta}g_{\alpha\beta} = 1 + 8J. \end{aligned} \tag{5.7}$$

It is well-known that P_{max} is simply square of larger Schmidt number in two-qubit case

$$P_{max} = \max\left(\lambda_0^2, \lambda_1^2\right). \tag{5.8}$$

It can be re-expressed in terms of reduced density operators

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4\det\rho^A}\right], \tag{5.9}$$

where $\rho^A = \mathrm{Tr}_B\rho = (1 + v_{1\alpha}\sigma_\alpha)/2$. Since P_{max} is invariant under LU-transformation, it should be expressed in terms of LU-invariant quantities. In fact, P_{max} in Eq.(5.9) can be re-written as

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4J}\right]. \tag{5.10}$$

Eq.(5.10) implies that P_{max} is manifestly LU-invariant.

5.2 Local Unitary Invariants

The Bloch representation of the 3-qubit density matrix can be written in the form

$$\begin{aligned}\rho = \frac{1}{8}\Bigg[\quad & \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + v_{1\alpha}\sigma_\alpha\otimes\mathbb{1}\otimes\mathbb{1} + v_{2\alpha}\mathbb{1}\otimes\sigma_\alpha\otimes\mathbb{1} + v_{3\alpha}\mathbb{1}\otimes\mathbb{1}\otimes\sigma_\alpha \\ & + h^{(1)}_{\alpha\beta}\mathbb{1}\otimes\sigma_\alpha\otimes\sigma_\beta + h^{(2)}_{\alpha\beta}\sigma_\alpha\otimes\mathbb{1}\otimes\sigma_\beta + h^{(3)}_{\alpha\beta}\sigma_\alpha\otimes\sigma_\beta\otimes\mathbb{1} \\ & + g_{\alpha\beta\gamma}\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_\gamma\Bigg],\end{aligned} \tag{5.11}$$

where σ_α is Pauli matrix. According to Eq.(5.6) and appendix A it is easy to show that the LU-invariants in the density matrix (5.11) are $|\vec{v}_1|$, $|\vec{v}_2|$, $|\vec{v}_3|$, $\mathrm{Tr}[h^{(1)}h^{(1)T}]$, $\mathrm{Tr}[h^{(2)}h^{(2)T}]$, $\mathrm{Tr}[h^{(3)}h^{(3)T}]$, $g_{\alpha\beta\gamma}g_{\alpha\beta\gamma}$ etc.

Few years ago Acín et al[68] represented the three-qubit arbitrary states in a simple form using a generalized Schmidt decomposition[121, 122] as following:

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{5.12}$$

with $\lambda_i \geq 0$, $0 \leq \varphi \leq \pi$, and $\sum_i \lambda_i^2 = 1$. The five algebraically independent polynomial LU-invariants were also constructed in Ref.[68]:

$$\begin{aligned} & J_1 = \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3\lambda_4\cos\varphi, \\ & J_2 = \lambda_0^2\lambda_2^2, \qquad J_3 = \lambda_0^2\lambda_3^2, \qquad J_4 = \lambda_0^2\lambda_4^2, \\ & J_5 = \lambda_0^2(J_1 + \lambda_2^2\lambda_3^2 - \lambda_1^2\lambda_4^2). \end{aligned} \tag{5.13}$$

In order to determine how many states have the same values of the invariants $J_1, J_2, ...J_5$, and therefore how many further discrete-valued invariants are needed to specify uniquely a pure state of three qubits up to local transformations, one would need to find the number of different sets of parameters φ and $\lambda_i (i = 0, 1, ...4)$, yielding the same invariants. Once λ_0 is found, other parameters are determined uniquely and therefore we derive an equation defining λ_0 in terms of polynomial invariants.

$$(J_1 + J_4)\lambda_0^4 - (J_5 + J_4)\lambda_0^2 + J_2 J_3 + J_2 J_4 + J_3 J_4 + J_4^2 = 0. \tag{5.14}$$

This equation has at most two positive roots and consequently an additional discrete-valued invariant is required to specify uniquely a pure three qubit state. Generally 18 LU-invariants, nine of which may be taken to have only discrete values, are needed to determine a mixed 2-qubit state [123].

If one represents the density matrix $|\psi\rangle\langle\psi|$ as a Bloch form like Eq.(5.11), it is possible to construct $v_{1\alpha}$, $v_{2\alpha}$, $v_{3\alpha}$, $h^{(1)}_{\alpha\beta}$, $h^{(2)}_{\alpha\beta}$, $h^{(3)}_{\alpha\beta}$, and $g_{\alpha\beta\gamma}$ explicitly, which are summarized in appendix B. Using these explicit expressions one can show directly that all polynomial LU-invariant quantities of pure states are expressed in terms of J_i as following:

$$\begin{aligned}
&|\vec{v}_1|^2 = 1 - 4(J_2 + J_3 + J_4), \qquad && |\vec{v}_2|^2 = 1 - 4(J_1 + J_3 + J_4) \\
&|\vec{v}_3|^2 = 1 - 4(J_1 + J_2 + J_4), && \mathrm{Tr}[h^{(1)}h^{(1)T}] = 1 + 4(2J_1 - J_2 - J_3) \\
&\mathrm{Tr}[h^{(2)}h^{(2)T}] = 1 - 4(J_1 - 2J_2 + J_3), && \mathrm{Tr}[h^{(3)}h^{(3)T}] = 1 - 4(J_1 + J_2 - 2J_3) \\
&g_{\alpha\beta\gamma}g_{\alpha\beta\gamma} = 1 + 4(2J_1 + 2J_2 + 2J_3 + 3J_4) && \\
&h^{(3)}_{\alpha\beta}v^{(1)}_\alpha v^{(2)}_\beta = 1 - 4(J_1 + J_2 + J_3 + J_4 - J_5). &&
\end{aligned} \tag{5.15}$$

Recently, Ref.[64] has shown that P_{max} for n-qubit pure state can be computed from $(n-1)$-qubit reduced mixed state. This is followed from a fact

$$\max_{R^1, R^2 \cdots R^n} \mathrm{Tr}\left[\rho R^1 \otimes R^2 \otimes \cdots \otimes R^n\right] = \max_{R^1, R^2 \cdots R^{n-1}} \mathrm{Tr}\left[\rho R^1 \otimes R^2 \otimes \cdots \otimes R^{n-1} \otimes \mathbb{1}\right] \tag{5.16}$$

which is Theorem I of Ref.[64]. Here, we would like to discuss the physical meaning of Eq.(5.16) from the aspect of LU-invariance. Eq.(5.16) in 3-qubit system reduces to

$$P_{max} = \max_{R^1, R^2} \mathrm{Tr}\left[\rho^{AB} R^1 \otimes R^2\right] \tag{5.17}$$

where $\rho^{AB} = \mathrm{Tr}_C \rho$. From Eq.(5.11) ρ^{AB} simply reduces to

$$\rho = \frac{1}{4}\left[\mathbb{1} \otimes \mathbb{1} + v_{1\alpha}\sigma_\alpha \otimes \mathbb{1} + v_{2\alpha}\mathbb{1} \otimes \sigma_\alpha + h^{(3)}_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta\right] \tag{5.18}$$

where $v_{1\alpha}$, $v_{2\alpha}$ and $h^{(3)}_{\alpha\beta}$ are explicitly given in appendix B. Of course, the LU-invariant quantities of ρ^{AB} are $|\vec{v}_1|$, $|\vec{v}_2|$, $\mathrm{Tr}[h^{(3)}h^{(3)T}]$, $h^{(3)}_{\alpha\beta}v_{1\alpha}v_{2\beta}$ etc, all of which, of course, can be re-expressed in terms of J_1, J_2, J_3, J_4 and J_5. It is worthwhile noting that we need all J_i's to express the LU-invariant quantities of ρ^{AB}. This means that the reduced state ρ^{AB} does have full information on the LU-invariance of the original pure state ρ.

Indeed, any reduced state resulting from a partial trace over a single qubit uniquely determines any entanglement measure of original system, given that the initial state is pure. Consider an $(n-1)$-qubit reduced density matrix that can be purified by a single qubit reference system. Let $|\psi'\rangle$ be any joint pure state. All other purifications can be obtained from the state $|\psi'\rangle$ by LU-transformations $U \otimes \mathbb{1}^{\otimes(n-1)}$, where U is a local unitary matrix acting on single qubit. Since any entanglement measure must be invariant under LU-transformations, it must be same for all purifications independently of U. Hence the reduced density matrix determines any entanglement measure on the initial pure state. That is why we can compute P_{max} of n-qubit pure state from the $(n-1)$-qubit reduced mixed state.

Generally, the information on the LU-invariance of the original n-qubit state is partly lost if we take partial trace twice. In order to show this explicitly let us consider $\rho^A \equiv \mathrm{Tr}_B\rho^{AB}$ and $\rho^B \equiv \mathrm{Tr}_A\rho^{AB}$:

$$\begin{aligned} \rho^A &= \frac{1}{2}\left[\mathbb{1} + v_{1\alpha}\sigma_\alpha\right] \\ \rho^B &= \frac{1}{2}\left[\mathbb{1} + v_{2\alpha}\sigma_\alpha\right]. \end{aligned} \tag{5.19}$$

Eq.(5.6) and appendix A imply that their LU-invariant quantities are only $|\vec{v}_1|$ and $|\vec{v}_2|$ respectively. Thus, we do not need J_5 to express the LU-invariant quantities of ρ^A and ρ^B. This fact indicates that the mixed states ρ^A and ρ^B partly loose the information of the LU-invariance of the original pure state ρ. This is why $(n-2)$-qubit reduced state cannot be used to compute P_{max} of n-qubit pure state.

5.3 Calculation of P_{max}

5.3.1 General Feature

If we insert the Bloch representation

$$R^1 = \frac{\mathbb{1} + \vec{s}_1\cdot\vec{\sigma}}{2} \qquad R^2 = \frac{\mathbb{1} + \vec{s}_2\cdot\vec{\sigma}}{2} \tag{5.20}$$

with $|\vec{s}_1| = |\vec{s}_2| = 1$ into Eq.(5.17), P_{max} for 3-qubit state becomes

$$P_{max} = \frac{1}{4} \max_{|\vec{s}_1|=|\vec{s}_2|=1} \left[1 + \vec{r}_1 \cdot \vec{s}_1 + \vec{r}_2 \cdot \vec{s}_2 + g_{ij} s_{1i} s_{2j}\right] \tag{5.21}$$

where

$$\begin{aligned} \vec{r}_1 &= \text{Tr}\left[\rho^A \vec{\sigma}\right] \\ \vec{r}_2 &= \text{Tr}\left[\rho^B \vec{\sigma}\right] \\ g_{ij} &= \text{Tr}\left[\rho^{AB} \sigma_i \otimes \sigma_j\right]. \end{aligned} \tag{5.22}$$

Since in Eq.(5.21) P_{max} is maximization with constraint $|\vec{s}_1| = |\vec{s}_2| = 1$, we should use the Lagrange multiplier method, which yields a pair of equations

$$\begin{aligned} \vec{r}_1 + g\vec{s}_2 &= \Lambda_1 \vec{s}_1 \\ \vec{r}_2 + g^T \vec{s}_1 &= \Lambda_2 \vec{s}_2, \end{aligned} \tag{5.23}$$

where the symbol g represents the matrix g_{ij} in Eq.(5.22). Thus we should solve $\vec{s}_1$, $\vec{s}_2$, Λ_1 and Λ_2 by eq.(5.23) and the constraint $|\vec{s}_1| = |\vec{s}_2| = 1$. Although it is highly nontrivial to solve Eq.(5.23), sometimes it is not difficult if the given 3-qubit state $|\psi\rangle$ has rich symmetries. Now, we would like to compute P_{max} for various types of 3-qubit system.

5.3.2 Type 1 (Product States): $J_1 = J_2 = J_3 = J_4 = J_5 = 0$

In order for all J_i's to be zero we have two cases $\lambda_0 = J_1 = 0$ or $\lambda_2 = \lambda_3 = \lambda_4 = 0$.

$\lambda_0 = J_1 = 0$

If $\lambda_0 = 0$, $|\psi\rangle$ in Eq.(5.12) becomes $|\psi\rangle = |1\rangle \otimes |BC\rangle$ where

$$|BC\rangle = \lambda_1 e^{i\varphi}|00\rangle + \lambda_2|01\rangle + \lambda_3|10\rangle + \lambda_4|11\rangle. \tag{5.24}$$

Thus P_{max} for $|\psi\rangle$ equals to that for $|BC\rangle$. Since $|BC\rangle$ is two-qubit state, one can easily compute P_{max} using Eq.(5.9), which is

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4\det\left(\text{Tr}_B|BC\rangle\langle BC|\right)}\right] = \frac{1}{2}\left[1 + \sqrt{1 - 4J_1}\right]. \tag{5.25}$$

If, therefore, $\lambda_0 = J_1 = 0$, we have $P_{max} = 1$, which gives a vanishing Groverian measure.

$\lambda_2 = \lambda_3 = \lambda_4 = 0$

In this case $|\psi\rangle$ in Eq.(5.12) becomes

$$|\psi\rangle = \big(\lambda_0|0\rangle + \lambda_1 e^{i\varphi}|1\rangle\big) \otimes |0\rangle \otimes |0\rangle. \tag{5.26}$$

Since $|\psi\rangle$ is completely product state, P_{max} becomes one.

5.3.3 Type2a (biseparable states)

In this type we have following three cases.

$J_1 \neq 0$ **and** $J_2 = J_3 = J_4 = J_5 = 0$

In this case we have $\lambda_0 = 0$. Thus P_{max} for this case is exactly same with Eq.(5.25).

$J_2 \neq 0$ **and** $J_1 = J_3 = J_4 = J_5 = 0$

In this case we have $\lambda_2 = \lambda_4 = 0$. Thus P_{max} for $|\psi\rangle$ equals to that for $|AC\rangle$, where

$$|AC\rangle = \lambda_0|00\rangle + \lambda_1 e^{i\varphi}|10\rangle + \lambda_2|11\rangle. \tag{5.27}$$

Using Eq.(5.9), therefore, one can easily compute P_{max}, which is

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4J_2}\right]. \tag{5.28}$$

$J_3 \neq 0$ **and** $J_1 = J_2 = J_4 = J_5 = 0$

In this case P_{max} for $|\psi\rangle$ equals to that for $|AB\rangle$, where

$$|AB\rangle = \lambda_0|00\rangle + \lambda_1 e^{i\varphi}|10\rangle + \lambda_3|11\rangle. \tag{5.29}$$

Thus P_{max} for $|\psi\rangle$ is

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4J_3}\right]. \tag{5.30}$$

5.3.4 Type2b (generalized GHZ states): $J_4 \neq 0$, $J_1 = J_2 = J_3 = J_5 = 0$

In this case we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $|\psi\rangle$ becomes

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_4|111\rangle \tag{5.31}$$

with $\lambda_0^2 + \lambda_4^2 = 1$. Then it is easy to show

$$\vec{r}_1 = \text{Tr}\left[\rho^A \vec{\sigma}\right] = (0, 0, \lambda_0^2 - \lambda_4^2) \tag{5.32}$$
$$\vec{r}_2 = \text{Tr}\left[\rho^B \vec{\sigma}\right] = (0, 0, \lambda_0^2 - \lambda_4^2)$$
$$g_{ij} = \text{Tr}\left[\rho^{AB} \sigma_i \otimes \sigma_j\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus P_{max} reduces to

$$P_{max} = \frac{1}{4} \max_{|\vec{s}_1| = |\vec{s}_2| = 1} \left[1 + (\lambda_0^2 - \lambda_4^2)(s_{1z} + s_{2z}) + s_{1z} s_{2z}\right]. \tag{5.33}$$

Since Eq.(5.33) is simple, we do not need to solve Eq.(5.23) for the maximization. If $\lambda_0 > \lambda_4$, the maximization can be achieved by simply choosing $\vec{s}_1 = \vec{s}_2 = (0, 0, 1)$. If $\lambda_0 < \lambda_4$, we choose $\vec{s}_1 = \vec{s}_2 = (0, 0, -1)$. Thus we have

$$P_{max} = \max(\lambda_0^2, \lambda_4^2). \tag{5.34}$$

In order to express P_{max} in Eq.(5.34) in terms of LU-invariants we follow the following procedure. First we note

$$P_{max} = \frac{1}{2}\left[(\lambda_0^2 + \lambda_4^2) + |\lambda_0^2 - \lambda_4^2|\right]. \tag{5.35}$$

Since $|\lambda_0^2 - \lambda_4^2| = \sqrt{(\lambda_0^2 + \lambda_4^2)^2 - 4\lambda_0^2 \lambda_4^2} = \sqrt{1 - 4J_4}$, we get finally

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4J_4}\right]. \tag{5.36}$$

5.3.5 Type3a (tri-Bell states)

In this case we have $\lambda_1 = \lambda_4 = 0$ and $|\psi\rangle$ becomes

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle \tag{5.37}$$

with $\lambda_0^2 + \lambda_2^2 + \lambda_3^2 = 1$. If we take LU-transformation σ_x in the first-qubit, $|\psi\rangle$ is changed into $|\psi'\rangle$ which is usual W-type state[65] as follows:

$$|\psi'\rangle = \lambda_0|100\rangle + \lambda_3|010\rangle + \lambda_2|001\rangle. \tag{5.38}$$

The LU-invariants in this type are

$$J_1 = \lambda_2^2 \lambda_3^2 \qquad J_2 = \lambda_0^2 \lambda_2^2 \tag{5.39}$$
$$J_3 = \lambda_0^2 \lambda_3^2 \qquad J_5 = 2\lambda_0^2 \lambda_2^2 \lambda_3^2.$$

Then it is easy to derive a relation

$$J_1 J_2 + J_1 J_3 + J_2 J_3 = \sqrt{J_1 J_2 J_3} = \frac{1}{2} J_5. \tag{5.40}$$

Recently, P_{max} for $|\psi'\rangle$ is computed analytically in Ref.[66] by solving the Lagrange multiplier equations (5.23) explicitly. In order to express P_{max} explicitly we first define

$$\begin{aligned} r_1 &= \lambda_3^2 + \lambda_2^2 - \lambda_0^2 \\ r_2 &= \lambda_0^2 + \lambda_2^2 - \lambda_3^2 \\ r_3 &= \lambda_0^2 + \lambda_3^2 - \lambda_2^2 \\ \omega &= 2\lambda_0 \lambda_3. \end{aligned} \tag{5.41}$$

Also we define

$$\begin{aligned} a &= \max(\lambda_0, \lambda_2, \lambda_3) \\ b &= \mathrm{mid}(\lambda_0, \lambda_2, \lambda_3) \\ c &= \min(\lambda_0, \lambda_2, \lambda_3). \end{aligned} \tag{5.42}$$

Then P_{max} is expressed differently in two different regions as follows. If $a^2 \geq b^2 + c^2$, P_{max} becomes

$$P_{max}^{>} = a^2 = \max(\lambda_0^2, \lambda_2^2, \lambda_3^2). \tag{5.43}$$

In order to express P_{max} in terms of LU-invariants we express Eq.(5.43) differently as

$$P_{max}^{>} = \frac{1}{4}\left[(\lambda_0^2 + \lambda_3^2 + \lambda_2^2) + |\lambda_0^2 + \lambda_3^2 - \lambda_2^2| + |\lambda_0^2 - \lambda_3^2 + \lambda_2^2| + |\lambda_0^2 - \lambda_3^2 - \lambda_2^2|\right]. \tag{5.44}$$

Using equalities

$$\begin{aligned} |\lambda_0^2 + \lambda_3^2 - \lambda_2^2| &= \sqrt{1 - 4\lambda_0^2\lambda_2^2 - 4\lambda_2^2\lambda_3^2} = \sqrt{1 - 4(J_1 + J_2)} \\ |\lambda_0^2 - \lambda_3^2 + \lambda_2^2| &= \sqrt{1 - 4\lambda_0^2\lambda_3^2 - 4\lambda_2^2\lambda_3^2} = \sqrt{1 - 4(J_1 + J_3)} \\ |\lambda_0^2 - \lambda_3^2 - \lambda_2^2| &= \sqrt{1 - 4\lambda_0^2\lambda_2^2 - 4\lambda_0^2\lambda_3^2} = \sqrt{1 - 4(J_2 + J_3)}, \end{aligned} \tag{5.45}$$

we can express P_{max} in Eq.(5.43) as follows:

$$P_{max}^{>} = \frac{1}{4}\left[1 + \sqrt{1 - 4(J_1 + J_2)} + \sqrt{1 - 4(J_1 + J_3)} + \sqrt{1 - 4(J_2 + J_3)}\right]. \tag{5.46}$$

If $a^2 \leq b^2 + c^2$, P_{max} becomes

$$P_{max}^{<} = \frac{1}{4}\left[1 + \frac{\omega\sqrt{(\omega^2 + r_1^2 - r_3^2)(\omega^2 + r_2^2 - r_3^2)} - r_1 r_2 r_3}{\omega^2 - r_3^2}\right]. \tag{5.47}$$

It was shown in Ref.[66] that $P_{max} = 4R^2$, where R is a circumradius of the triangle λ_0, λ_2 and λ_3. When $a^2 \leq b^2 + c^2$, one can show easily $r_1 = \sqrt{1 - 4(J_2 + J_3)}$, $r_2 = \sqrt{1 - 4(J_1 + J_3)}$, $r_3 = \sqrt{1 - 4(J_1 + J_2)}$, and $\omega = 2\sqrt{J_3}$. Using $\omega^2 - r_3^2 - r_1 r_2 r_3 = 8\lambda_0^2\lambda_2^2\lambda_3^2$, One can show easily that P_{max} in Eq.(5.47) in terms of LU-invariants becomes

$$P_{max}^{<} = \frac{4\sqrt{J_1 J_2 J_3}}{4(J_1 + J_2 + J_3) - 1}. \tag{5.48}$$

Let us consider $\lambda_0 = 0$ limit in this type. Then we have $J_2 = J_3 = 0$. Thus $P_{max}^{>}$ reduces to $(1/2)(1 + \sqrt{1 - 4J_1})$ which exactly coincides with Eq.(5.25). By same way one can prove that Eq.(5.46) has correct limits to various other types.

5.3.6 Type3b (extended GHZ states)

This type consists of 3 types, *i.e.* $\lambda_1 = \lambda_2 = 0$, $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 = \lambda_3 = 0$.

$\lambda_1 = \lambda_2 = 0$

In this case the state (5.12) becomes

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{5.49}$$

with $\lambda_0^2 + \lambda_3^2 + \lambda_4^2 = 1$. The non-vanishing LU-invariants are

$$J_3 = \lambda_0^2\lambda_3^2, \qquad J_4 = \lambda_0^2\lambda_4^2. \tag{5.50}$$

Note that $J_3 + J_4$ is expressed in terms of solely λ_0 as

$$J_3 + J_4 = \lambda_0^2(1 - \lambda_0^2). \tag{5.51}$$

Eq.(5.49) can be re-written as

$$|\psi\rangle = \lambda_0|00q_1\rangle + \sqrt{1 - \lambda_0^2}|11q_2\rangle \tag{5.52}$$

where $|q_1\rangle = |0\rangle$ and $|q_2\rangle = (1/\sqrt{1 - \lambda_0^2})(\lambda_3|0\rangle + \lambda_4|1\rangle)$ are normalized one qubit states. Thus, from Ref.[66], P_{max} for $|\psi\rangle$ is

$$P_{max} = \max\left(\lambda_0^2, 1 - \lambda_0^2\right) = \frac{1}{2}\left[1 + \sqrt{(1 - 2\lambda_0^2)^2}\right]. \tag{5.53}$$

With an aid of Eq.(5.51) P_{max} in Eq.(5.53) can be easily expressed in terms of LU-invariants as following:

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4(J_3 + J_4)}\right]. \tag{5.54}$$

If we take $\lambda_3 = 0$ limit in this type, we have $J_3 = 0$, which makes Eq.(5.54) to be $(1/2)(1 + \sqrt{1 - 4J_4})$. This exactly coincides with Eq.(5.36).

$\lambda_1 = \lambda_3 = 0$

In this case $|\psi\rangle$ and LU-invariants are

$$|\psi\rangle = \lambda_0|0q_1 0\rangle + \sqrt{1 - \lambda_0^2}|1q_2 1\rangle \tag{5.55}$$

and

$$J_2 = \lambda_0^2\lambda_2^2, \qquad J_4 = \lambda_0^2\lambda_4^2 \tag{5.56}$$

where $|q_1\rangle = |0\rangle$, $|q_2\rangle = (1/\sqrt{1 - \lambda_0^2})(\lambda_2|0\rangle + \lambda_4|1\rangle)$, and $\lambda_0^2 + \lambda_2^2 + \lambda_4^2 = 1$. The same method used in the previous subsection easily yields

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4(J_2 + J_4)}\right]. \tag{5.57}$$

One can show that Eq.(5.57) has correct limits to other types.

$\lambda_2 = \lambda_3 = 0$

In this case $|\psi\rangle$ and LU-invariants are

$$|\psi\rangle = \sqrt{1 - \lambda_4^2}|q_1 00\rangle + \lambda_4|q_2 11\rangle \tag{5.58}$$

and

$$J_1 = \lambda_1^2\lambda_4^2, \qquad J_4 = \lambda_0^2\lambda_4^2 \tag{5.59}$$

where $|q_1\rangle = (1/\sqrt{1 - \lambda_4^2})(\lambda_0|0\rangle + \lambda_1 e^{i\varphi}|1\rangle)$, $|q_2\rangle = |1\rangle$, and $\lambda_0^2 + \lambda_1^2 + \lambda_4^2 = 1$. It is easy to show

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4(J_1 + J_4)}\right]. \tag{5.60}$$

One can show that Eq.(5.60) has correct limits to other types.

5.3.7 Type4a ($\lambda_4 = 0$)

In this case the state vector $|\psi\rangle$ in Eq.(5.12) reduces to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle \tag{5.61}$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. The non-vanishing LU-invariants are

$$\begin{aligned} J_1 &= \lambda_2^2\lambda_3^2 \qquad & J_2 &= \lambda_0^2\lambda_2^2 \\ J_3 &= \lambda_0^2\lambda_3^2 \qquad & J_5 &= 2\lambda_0^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{5.62}$$

From Eq.(5.62) it is easy to show

$$\sqrt{J_1 J_2 J_3} = \frac{1}{2} J_5. \tag{5.63}$$

The remarkable fact deduced from Eq.(5.62) is that the non-vanishing LU-invariants are independent of the phase factor φ. This indicates that the Groverian measure for Eq.(5.61) is also independent of φ

In order to compute P_{max} analytically in this type, we should solve the Lagrange multiplier equations (5.23) with

$$\begin{aligned} \vec{r}_1 &= \mathrm{Tr}[\rho^A \vec{\sigma}] = (2\lambda_0\lambda_1\cos\varphi, 2\lambda_0\lambda_1\sin\varphi, 2\lambda_0^2 - 1) \\ \vec{r}_2 &= \mathrm{Tr}[\rho^B \vec{\sigma}] = (2\lambda_1\lambda_3\cos\varphi, -2\lambda_1\lambda_3\sin\varphi, 1 - 2\lambda_3^2) \\ g_{ij} &= \mathrm{Tr}[\rho^{AB}\sigma_i \otimes \sigma_j] = \begin{pmatrix} 2\lambda_0\lambda_3 & 0 & 2\lambda_0\lambda_1\cos\varphi \\ 0 & -2\lambda_0\lambda_3 & 2\lambda_0\lambda_1\sin\varphi \\ -2\lambda_1\lambda_3\cos\varphi & 2\lambda_1\lambda_3\sin\varphi & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{pmatrix}. \end{aligned} \tag{5.64}$$

Although we have freedom to choose the phase factor φ, it is impossible to find singular values of the matrix g, which makes it formidable task to solve Eq.(5.23). Based on Ref.[66] and Ref.[67], furthermore, we can conjecture that P_{max} for this type may have several different expressions depending on the domains in parameter space. Therefore, it may need long calculation to compute P_{max} analytically. We would like to leave this issue for our future research work and the explicit expressions of P_{max} are not presented in this chapter.

5.3.8 Type4b

This type consists of the 2 cases, *i.e.* $\lambda_2 = 0$ or $\lambda_3 = 0$.

$\lambda_2 = 0$

In this case the state vector $|\psi\rangle$ in Eq.(5.12) reduces to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{5.65}$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_3^2 + \lambda_4^2 = 1$. The LU-invariants are

$$J_1 = \lambda_1^2\lambda_4^2 \qquad J_3 = \lambda_0^2\lambda_3^2 \qquad J_4 = \lambda_0^2\lambda_4^2. \tag{5.66}$$

Eq.(5.66) implies that the Groverian measure for Eq.(5.65) is independent of the phase factor φ like type 4a. This fact may drastically reduce the calculation procedure for solving the Lagrange multiplier equation (5.23). In spite of this fact, however, solving Eq.(5.23) is highly non-trivial as we commented in the previous type. The explicit expressions of the Groverian measure are not presented in this chapter and we hope to present them elsewhere in the near future.

$\lambda_3 = 0$

In this case the state vector $|\psi\rangle$ in Eq.(5.12) reduces to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle \tag{5.67}$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_4^2 = 1$. The LU-invariants are

$$J_1 = \lambda_1^2\lambda_4^2 \qquad J_2 = \lambda_0^2\lambda_2^2 \qquad J_4 = \lambda_0^2\lambda_4^2. \tag{5.68}$$

Eq.(5.68) implies that the Groverian measure for Eq.(5.67) is independent of the phase factor φ like type 4a.

5.3.9 Type4c ($\lambda_1 = 0$)

In this case the state vector $|\psi\rangle$ in Eq.(5.12) reduces to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{5.69}$$

with $\lambda_0^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$. The LU-invariants in this type are

$$\begin{aligned} &J_1 = \lambda_2^2\lambda_3^2 \qquad J_2 = \lambda_0^2\lambda_2^2 \qquad J_3 = \lambda_0^2\lambda_3^2 \\ &J_4 = \lambda_0^2\lambda_4^2 \qquad J_5 = 2\lambda_0^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{5.70}$$

From Eq.(5.70) it is easy to show

$$J_1(J_2 + J_3 + J_4) + J_2 J_3 = \sqrt{J_1 J_2 J_3} = \frac{1}{2} J_5. \tag{5.71}$$

In this type $\vec{r}_1$, $\vec{r}_2$ and g_{ij} defined in Eq.(5.22) are

$$\vec{r}_1 = (0, 0, 2\lambda_0^2 - 1) \tag{5.72}$$
$$\vec{r}_2 = (2\lambda_2\lambda_4, 0, \lambda_0^2 + \lambda_2^2 - \lambda_3^3 - \lambda_4^2)$$
$$g_{ij} = \begin{pmatrix} 2\lambda_0\lambda_3 & 0 & 0 \\ 0 & -2\lambda_0\lambda_3 & 0 \\ -2\lambda_2\lambda_4 & 0 & 1 - 2\lambda_2^2 \end{pmatrix}.$$

Like type 4a and type 4b solving Eq.(5.23) is highly non-trivial mainly due to non-diagonalization of g_{ij}. Of course, the fact that the first component of $\vec{r}_2$ is non-zero makes hard to solve Eq.(5.23) too. The explicit expressions of the Groverian measure in this type are not given in this chapter.

5.3.10 Type5 (real states): $\varphi = 0$, π

$\varphi = 0$

In this case the state vector $|\psi\rangle$ in Eq.(5.12) reduces to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{5.73}$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$. The LU-invariants in this case are

$$J_1 = (\lambda_2\lambda_3 - \lambda_1\lambda_4)^2 \quad J_2 = \lambda_0^2\lambda_2^2 \quad J_3 = \lambda_0^2\lambda_3^2 \tag{5.74}$$
$$J_4 = \lambda_0^2\lambda_4^2 \quad J_5 = 2\lambda_0^2\lambda_2\lambda_3(\lambda_2\lambda_3 - \lambda_1\lambda_4).$$

It is easy to show $\sqrt{J_1J_2J_3} = J_5/2$.

$\varphi = \pi$

In this case the state vector $|\psi\rangle$ in Eq.(5.12) reduces to

$$|\psi\rangle = \lambda_0|000\rangle - \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{5.75}$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$. The LU-invariants in this case are

$$J_1 = (\lambda_2\lambda_3 + \lambda_1\lambda_4)^2 \quad J_2 = \lambda_0^2\lambda_2^2 \quad J_3 = \lambda_0^2\lambda_3^2 \tag{5.76}$$
$$J_4 = \lambda_0^2\lambda_4^2 \quad J_5 = 2\lambda_0^2\lambda_2\lambda_3(\lambda_2\lambda_3 + \lambda_1\lambda_4).$$

It is easy to show $\sqrt{J_1J_2J_3} = J_5/2$ in this type.

The analytic calculation of P_{max} in type 5 is most difficult problem. In addition, we don't know whether it is mathematically possible or not. However, the geometric

Type		conditions	P_{max}
Type I		$J_i = 0$	1
Type II	a	$J_i = 0$ except J_1	$\frac{1}{2}\left(1+\sqrt{1-4J_1}\right)$
		$J_i = 0$ except J_2	$\frac{1}{2}\left(1+\sqrt{1-4J_2}\right)$
		$J_i = 0$ except J_3	$\frac{1}{2}\left(1+\sqrt{1-4J_3}\right)$
	b	$J_i = 0$ except J_4	$\frac{1}{2}\left(1+\sqrt{1-4J_4}\right)$
Type III	a	$\lambda_1 = \lambda_4 = 0$	$\frac{1}{4}\left(1+\sqrt{1-4(J_1+J_2)}+\sqrt{1-4(J_1+J_3)}+\sqrt{1-4(J_2+J_3)}\right)$ if $a^2 \geq b^2 + c^2$
			$4\sqrt{J_1 J_2 J_3}/\left(4(J_1+J_2+J_3)-1\right)$ if $a^2 \leq b^2 + c^2$
	b	$\lambda_1 = \lambda_2 = 0$	$\frac{1}{2}\left(1+\sqrt{1-4(J_3+J_4)}\right)$
		$\lambda_1 = \lambda_3 = 0$	$\frac{1}{2}\left(1+\sqrt{1-4(J_2+J_4)}\right)$
		$\lambda_2 = \lambda_3 = 0$	$\frac{1}{2}\left(1+\sqrt{1-4(J_1+J_4)}\right)$
Type IV	a	$\lambda_4 = 0$	independent of φ: not presented
	b	$\lambda_2 = 0$	independent of φ: not presented
		$\lambda_3 = 0$	independent of φ: not presented
	c	$\lambda_1 = 0$	not presented
Type V		$\varphi = 0$	not presented
		$\varphi = \pi$	not presented

Table 5.1: Summary of P_{max} in various types.

interpretation of P_{max} presented in Ref.[66] and Ref.[67] may provide us valuable insight. We hope to leave this issue for our future research work too. The results in this section is summarized in Table I.

5.4 New Type

5.4.1 standard form

In this section we consider new type in 3-qubit states. The type we consider is

$$|\Phi\rangle = a|100\rangle + b|010\rangle + c|001\rangle + q|111\rangle, \quad a^2 + b^2 + c^2 + q^2 = 1. \tag{5.77}$$

First, we would like to derive the standard form like Eq.(5.12) from $|\Phi\rangle$. This can be achieved as following. First, we consider LU-transformation of $|\Phi\rangle$, *i.e.* $(U \otimes \mathbb{1} \otimes \mathbb{1})|\Phi\rangle$,

where

$$U = \frac{1}{\sqrt{aq+bc}} \begin{pmatrix} \sqrt{aq}e^{i\theta} & \sqrt{bc}e^{i\theta} \\ -\sqrt{bc} & \sqrt{aq} \end{pmatrix}. \tag{5.78}$$

After LU-transformation, we perform Schmidt decomposition following Ref.[68]. Finally we choose θ to make all λ_i to be positive. Then we can derive the standard form (5.12) from $|\Phi\rangle$ with $\varphi = 0$ or π, and

$$\begin{aligned} \lambda_0 &= \sqrt{\frac{(ac+bq)(ab+cq)}{aq+bc}} \\ \lambda_1 &= \frac{\sqrt{abcq}}{\sqrt{(ab+cq)(ac+bq)(aq+bc)}}|a^2+q^2-b^2-c^2| \\ \lambda_2 &= \frac{1}{\lambda_0}|ac-bq| \\ \lambda_3 &= \frac{1}{\lambda_0}|ab-cq| \\ \lambda_4 &= \frac{2\sqrt{abcq}}{\lambda_0}. \end{aligned} \tag{5.79}$$

It is easy to prove that the normalization condition $a^2+b^2+c^2+q^2=1$ guarantees the normalization

$$\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2 = 1. \tag{5.80}$$

Since $|\Phi\rangle$ has three free parameters, we need one more constraint between λ_i's. This additional constraint can be derived by trial and error. The explicit expression for this additional relation is

$$\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2) = \frac{1}{4} - \frac{\lambda_1^2}{\lambda_4^2}(\lambda_2^2+\lambda_4^2)(\lambda_3^2+\lambda_4^2). \tag{5.81}$$

Since all λ_i's are not vanishing but there are only three free parameters, $|\Phi\rangle$ is not involved in the types discussed in the previous section.

5.4.2 LU-invariants

Using Eq.(5.79) it is easy to derive LU-invariants which are

$$
\begin{aligned}
J_1 &= (\lambda_1\lambda_4 - \lambda_2\lambda_3)^2 = \frac{1}{(ab+cq)^2(ac+bq)^2} \\
&\qquad \times \left[2abcq|a^2+q^2-b^2-c^2| - (aq+bc)|(ab-cq)(ac-bq)|\right]^2 \\
J_2 &= \lambda_0^2\lambda_2^2 = (ac-bq)^2 \\
J_3 &= \lambda_0^2\lambda_3^2 = (ab-cq)^2 \\
J_4 &= \lambda_0^2\lambda_4^2 = 4abcq \\
J_5 &= \lambda_0^2\left(J_1 + \lambda_2^2\lambda_3^2 - \lambda_1^2\lambda_4^2\right).
\end{aligned}
\tag{5.82}
$$

One can show directly that $J_5 = 2\sqrt{J_1J_2J_3}$. Since $|\Phi\rangle$ has three free parameters, there should exist additional relation between J_i's. However, the explicit expression may be hardly derived. In principle, this constraint can be derived as following. First, we express the coefficients a, b, c, and q in terms of J_1, J_2, J_3 and J_4 using first four equations of Eq.(5.82). Then the normalization condition $a^2+b^2+c^2+q^2=1$ gives explicit expression of this additional constraint. Since, however, this procedure requires the solutions of quartic equation, it seems to be hard to derive it explicitly.

Since J_1 contains absolute value, it is dependent on the regions in the parameter space. Direct calculation shows that J_1 is

$$
J_1 = \begin{cases}
(aq-bc)^2 \qquad \text{when } (a^2+q^2-b^2-c^2)(ab-cq)(ac-bq) \geq 0 \\
(aq-bc)^2[1+2(ab-cq)(ac-bq)(aq+bc)/(ab+cq)(ac+bq)(aq-bc)]^2 \\
\qquad\qquad \text{when } (a^2+q^2-b^2-c^2)(ab-cq)(ac-bq) < 0.
\end{cases}
\tag{5.83}
$$

Since P_{max} is manifestly LU-invariant quantity, it is obvious that it also depends on the regions on the parameter space.

5.4.3 calculation of P_{max}

P_{max} for state $|\Phi\rangle$ in Eq.(5.77) has been analytically computed recently in Ref.[67]. It turns out that P_{max} is differently expressed in three distinct ranges of definition in parameter space. The final expressions can be interpreted geometrically as discussed in Ref.[67]. To express P_{max} explicitly we define

$$
\begin{aligned}
& r_1 \equiv b^2+c^2-a^2-q^2 \qquad r_2 \equiv a^2+c^2-b^2-q^2 \\
& r_3 \equiv a^2+b^2-c^2-q^2 \qquad \omega \equiv ab+qc \qquad \mu \equiv ab-qc.
\end{aligned}
\tag{5.84}
$$

The first expression of P_{max}, which can be expressed in terms of circumradius of convex quadrangle is

$$P_{max}^{(Q)} = \frac{4(ab+qc)(ac+qb)(aq+bc)}{4\omega^2 - r_3^2}. \tag{5.85}$$

The second expression of P_{max}, which can be expressed in terms of circumradius of crossed-quadrangle is

$$P_{max}^{(CQ)} = \frac{(ab-cq)(ac-bq)(bc-aq)}{4S_x^2} \tag{5.86}$$

where

$$S_x^2 = \frac{1}{16}(a+b+c+q)(a+b-c-q)(a-b+c-q)(-a+b+c-q). \tag{5.87}$$

The final expression of P_{max} corresponds to the largest coefficient:

$$P_{max}^{(L)} = \max(a^2, b^2, c^2, q^2) = \frac{1}{4}\left(1 + |r_1| + |r_2| + |r_3|\right). \tag{5.88}$$

The applicable domain for each P_{max} is fully discussed in Ref.[67].

Now we would like to express all expressions of P_{max} in terms of LU-invariants. For the simplicity we choose a simplified case, that is $(a^2+q^2-b^2-c^2)(ab-cq)(ac-bq) \geq 0$. Then it is easy to derive

$$\begin{aligned} r_1^2 &= 1 - 4(J_2+J_3+J_4) \qquad & r_2^2 &= 1 - 4(J_1+J_3+J_4) \\ r_3^2 &= 1 - 4(J_1+J_2+J_4) \qquad & \omega^2 &= J_3 + J_4. \end{aligned} \tag{5.89}$$

Then it is simple to express $P_{max}^{(Q)}$ and $P_{max}^{(CQ)}$ as following:

$$\begin{aligned} P_{max}^{(Q)} &= \frac{4\sqrt{(J_1+J_4)(J_2+J_4)(J_3+J_4)}}{4(J_1+J_2+J_3+2J_4)-1} \\ P_{max}^{(CQ)} &= \frac{4\sqrt{J_1J_2J_3}}{4(J_1+J_2+J_3+J_4)-1}. \end{aligned} \tag{5.90}$$

If we take $q = 0$ limit, we have $\lambda_4 = J_4 = 0$. Thus $P_{max}^{(Q)}$ and $P_{max}^{(CQ)}$ reduce to $4\sqrt{J_1J_2J_3}/(4(J_1+J_2+J_3)-1)$, which exactly coincides with $P_{max}^{<}$ in Eq.(5.48). Finally Eq.(5.89) makes $P_{max}^{(L)}$ to be

$$P_{max}^{(L)} = \frac{1}{4}\Big(1 + \sqrt{1-4(J_2+J_3+J_4)} + \sqrt{1-4(J_1+J_3+J_4)} + \sqrt{1-4(J_1+J_2+J_4)}\Big). \tag{5.91}$$

One can show that $P_{max}^{(L)}$ equals to $P_{max}^{>}$ in Eq.(5.46) when $q = 0$. This indicates that our results (5.90) and (5.91) have correct limits to other types of three-qubit system.

5.5 conclusion

We tried to compute the Groverian measure analytically in the various types of three-qubit system. The types we considered in this chapter are given in Ref.[68] for the classification of the three-qubit system.

For type 1, type 2 and type 3 the Groverian measures are analytically computed. All results, furthermore, can be represented in terms of LU-invariant quantities. This reflects the manifest LU-invariance of the Groverian measure.

For type 4 and type 5 we could not derive the analytical expressions of the measures because the Lagrange multiplier equations (5.23) is highly difficult to solve. However, the consideration of LU-invariants indicates that the Groverian measure in type 4 should be independent of the phase factor φ. We expect that this fact may drastically simplify the calculational procedure for obtaining the analytical results of the measure in type 4. The derivation in type 5 is most difficult problem. However, it might be possible to get valuable insight from the geometric interpretation of P_{max}, presented in Ref.[66] and Ref.[67]. We would like to revisit type 4 and type 5 in the near future.

We think that the most important problem in the research of entanglement is to understand the general properties of entanglement measures in arbitrary qubit systems. In order to explore this issue we would like to extend, as a next step, our calculation to four-qubit states. In addition, the Groverian measure for four-qubit pure state is related to that for two-qubit mixed state via purification[92]. Although general theory for entanglement is far from complete understanding at present stage, we would like to go toward this direction in the future.

Chapter 6

Toward an understanding of entanglement for generalized n-qubit W-states

Entanglement of quantum states [124] plays an important role in quantum information, computation and communication(QICC). It is a genuine physical resource for the teleportation process [19, 57] and makes it possible that the quantum computer outperforms classical one [32, 97]. It also plays a crucial role in quantum cryptographic schemes [21, 125]. These phenomena have provided the basis for the development of modern quantum information science.

Quantum entanglement is a rich field of research. A better understanding of quantum entanglement, of ways it is characterized, created, detected, stored and manipulated, is theoretically the most basic task of the current QICC research. In bipartite case entanglement is relatively well understood, while in multipartite case even quantifying entanglement of pure states is a great challenge.

The geometric measure of entanglement can be considered as one of the most reliable quantifiers of multipartite entanglement [41, 42, 43]. It depends on P_{max}, the maximal overlap of a given state with the nearest product state, and is defined by the formula $E_g(\psi) = 1 - P_{max}$ [43]. The same overlap P_{max}, known also as the injective tensor norm of ψ [111], is the maximal probability of success in the Grover's search algorithm [24] when the state ψ is used as an input state. This relationship between the success probability of the quantum search algorithm and the amount of entanglement of the input state allows oneself to define an operational entanglement measure known as Groverian entanglement [46, 93].

The maximal overlap P_{max} is a useful quantity and has several practical applications. It has been used to study quantum phase transitions in spin models [126, 127] and to quantify the distinguishability of multipartite states by local means [128]. Moreover, P_{max} exhibits interesting connections with entanglement witnesses and can be efficiently estimated in experiments [129]. Recently, it has been shown that the maximal overlap is the largest coefficient of the generalized Schmidt decomposition and the nearest product state uniquely defines the factorizable basis of the decomposition [130, 131].

In spite of its usefulness one obstacle to use P_{max} fully in quantum information theories is the fact that it is difficult to compute it analytically for generic states. The usual maximization method generates a system of nonlinear equations [43]. Thus, it is important to develop a technique for the computation of P_{max} [64, 132, 133, 134, 135].

Theorem I of Ref.[64] enables us to compute P_{max} for n-qubit pure states by making use of $(n-1)$-qubit reduced states. In the case of three-qubit states the theorem effectively changes the nonlinear eigenvalue equations into the linear form. Owing to this essential simplification P_{max} for the generalized three-qubit W-state [65, 136] was computed analytically in Ref.[66]. Furthermore, in Ref.[67] P_{max} was found for three-qubit quadrilateral states with an elegant geometric interpretation. More recently, based on the analytical results of Ref.[66, 67] and the classification of Ref.[68], P_{max} for various types of three-qubit states was computed analytically and expressed in terms of local unitary(LU) invariants [69].

In general, the calculation of the multi-partite entanglement is confronted with great difficulties. Furthermore, even if we know explicit expressions of entanglement measure, the separation of the applicable domains is also a nontrivial task [67]. Therefore, there is a good reason to consider first some solvable cases that allow analytic solutions and clear separations of the validity domains. Later, these results could be extended, either analytically or numerically, for a wider class of multi-qubit states. In the light of these ideas we consider one- and two-parametric n-qubit W-type states with $n \geq 4$ in this chapter.

The chapter is organized as follows. In Sec. 6.1 we clarify our tasks and notations. In Sec. 6.2 we review the calculational tool introduced in Ref.[64, 66, 67] and explain how the Lagrange multiplier method gives simple solution to the one-parameter cases. This method is used Sec. 6.3 for the derivation of P_{max} for one-parameter W-states in 4-qubit, 5-qubit and 6-qubit systems. In this section the analytical results are compared with numerical data. In Sec. 6.4 based on the analytical results of the previous section we compute P_{max} for an one-parameter W-state in arbitrary n-qubit system. In Sec. 6.5 we derive P_{max} for two-parameter W-states in 4-qubit system by adopting the usual maximization technique. In Sec. 6.6 we analyze two-parameter results by considering several particular cases. In Sec. 6.7 we discuss the possibility of extensions of the results to arbitrary W states and the existence of a geometrical interpretation.

6.1 Summary of Tasks

Let $|\psi\rangle$ be a pure state of an n-party system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where the dimensions of the individual state spaces $\mathcal{H}_k$ are finite but otherwise arbitrary. The maximal overlap of $|\psi\rangle$ is given by

$$P_{max}(\psi) \equiv \max_{|q_1\rangle\cdots|q_n\rangle} |\langle q_1|\langle q_2|\cdots\langle q_n|\psi\rangle|^2, \tag{6.1}$$

where the maximum is taken over all single-system normalized state vectors $|q_k\rangle \in \mathcal{H}_k$, and it is understood that $|\psi\rangle$ is normalized.

Let us consider now n-qubit W-type state

$$|W_n\rangle = a_1|10\cdots0\rangle + a_2|010\cdots0\rangle + \cdots + a_n|0\cdots01\rangle, \tag{6.2}$$

where the labels within each ket refer to qubits $1, 2, \cdots, n$ in that order.

In this chapter we will compute analytically P_{max} in the following two cases:

1)for the one-parametric $|W_n\rangle$ when $a_1 = \cdots = a_{n-1} \equiv a$ and $a_n \equiv q$

2)for the two-parametric $|W_4\rangle$ when $a_1 = a,\ a_2 = b,\ a_3 = a_4 = q$.

To ensure the calculational validity we use the result of [93], which has shown that $P_{max} = (1 - 1/n)^{n-1}$ when $a_1 = a_2 = \cdots = a_n$. Thus, the final results of the one-parametric case should agree with the following:

• If $a = q = 1/\sqrt{n}$, then P_{max} should be equal to $(1 - 1/n)^{n-1}$.

• If $q = 0$, then $|W_n\rangle$ becomes $|W_{n-1}\rangle \otimes |0\rangle$ and, as a result, P_{max} should be equal to $(1 - 1/(n-1))^{n-2}$.

For the two-parametric case $P_{max}(W_4)$ should have a correct limit when either a or b vanishes. At $a = 0$ we have $|W_4\rangle = |0\rangle \otimes |W_3\rangle$ and thus the maximal overlap should be expressed in terms of the circumradius of the isosceles triangle with sides b, q, q [66].

6.2 Calculation Tool

For a pure state of two qubits P_{max} is given by

$$P_{max} = \frac{1}{2}\left[1 + \sqrt{1 - 4\det\rho^A}\right], \tag{6.3}$$

where ρ^A is reduced density matrix, i.e. $\mathrm{Tr}_B\rho^{AB}$. Therefore, the Bell (and their LU-equivalent) states have the minimal ($P_{max} = 1/2$) while product states have the maximal ($P_{max} = 1$) overlap.

The explicit dependence of P_{max} on state parameters for the generalized three-qubit W-state

$$|W_3\rangle = a_1|100\rangle + a_2|010\rangle + a_3|001\rangle \tag{6.4}$$

was computed in Ref.[66]. In order to express explicitly $P_{max}(W_3)$ in terms of state parameters, we define a set $\{\alpha, \beta, \gamma\}$ as the set $\{a_1, a_2, a_3\}$ in decreasing order. Then P_{max} for the generalized W-state can be expressed in a form

$$P_{max}(W_3) = \begin{cases} 4R_W^2 & \text{when } \alpha^2 \leq \beta^2 + \gamma^2 \\ \alpha^2 & \text{when } \alpha^2 \geq \beta^2 + \gamma^2 \end{cases} \tag{6.5}$$

where R_W is the circumradius of the triangle with sides a_1, a_2, a_3. Similar calculation procedure can be applied to the 3-qubit quadrilateral state. It has been shown in Ref.[67] that for this case P_{max} is expressed in terms of the circumradius of a convex quadrangle. These two separate results strongly suggest that P_{max} for an arbitrary pure state has its own geometrical meaning. If we are able to know this meaning completely, then our understanding on the multipartite entanglement would be greatly enhanced.

Now, we briefly review how to derive the analytic result (6.5) because it plays crucial role in next two sections. In Ref.[66] P_{max} for 3-qubit state is expressed as

$$P_{max} = \frac{1}{4}\max_{|\vec{s}_1|=|\vec{s}_2|=1}\left[1 + \vec{s}_1\cdot\vec{r}_1 + \vec{s}_2\cdot\vec{r}_2 + g_{ij}s_{1i}s_{2j}\right] \tag{6.6}$$

where $\vec{s}_1$ and $\vec{s}_2$ are Bloch vectors of the single-qubit states. In Eq.(6.6) $\vec{r}_1 = Tr[\rho^A\vec{\sigma}]$, $\vec{r}_2 = Tr[\rho^B\vec{\sigma}]$ and $g_{ij} = Tr[\rho^{AB}\sigma_i\otimes\sigma_j]$, where ρ^A, ρ^B and ρ^{AB} are appropriate partial traces of $\rho^{ABC} \equiv |W_3\rangle\langle W_3|$ and σ_i are usual Pauli matrices. The explicit expressions of $\vec{r}_1$, $\vec{r}_2$ and g_{ij} are given in Ref.[66]. Due to maximization over $\vec{s}_1$ and $\vec{s}_2$ in Eq.(6.6) we can compute $\vec{s}_1$ and $\vec{s}_2$ by solving the Lagrange multiplier equations

$$\vec{r}_1 + g\vec{s}_2 = \lambda_1\vec{s}_1, \qquad \vec{r}_2 + g^T\vec{s}_1 = \lambda_2\vec{s}_2, \tag{6.7}$$

where λ_1 and λ_2 are Lagrange multiplier constants. Now, we let $s_{1y} = s_{2y} = 0$ for simplicity, because those give only irrelevant overall phase factor to $\langle q_1|\langle q_2|\langle q_3|W_3\rangle$. After eliminating the Lagrange multiplier constants, one can show that Eq.(6.7) reduces to two equations. Examining these two remaining equations, one can show that $\vec{s}_1$ and $\vec{s}_2$ have a following relation to each other:

$$\vec{s}_1(a_1, a_2, a_3) = \vec{s}_2(a_2, a_1, a_3). \tag{6.8}$$

Using this relation, one can combine these two equations into single one expressed in terms of solely s_{1z} in a final form

$$\frac{\sqrt{1 - s_{1z}^2(a_1, a_2, a_3)}}{s_{1z}(a_1, a_2, a_3)} = \frac{\omega\sqrt{1 - s_{1z}^2(a_2, a_1, a_3)}}{r_1 - r_3 s_{1z}(a_2, a_1, a_3)} \tag{6.9}$$

where $r_1 = a_2^2 + a_3^2 - a_1^2$, $r_2 = a_1^2 + a_3^2 - a_2^2$, $r_3 = a_1^2 + a_2^2 - a_3^2$ and $\omega = 2a_1a_2$. Defining $a_1 = a_2 \equiv a$ and $a_3 \equiv q$ again, one can solve Eq.(6.9) easily in a form

$$\begin{aligned} s_{1z} = s_{2z} &= \frac{r_1}{\omega + r_3} = \frac{q^2}{4a^2 - q^2} \\ s_{1x} = s_{2x} &= \sqrt{1 - s_{1z}^2} = \frac{2\sqrt{2}a}{4a^2 - q^2}\sqrt{2a^2 - q^2}. \end{aligned} \tag{6.10}$$

Inserting Eq.(6.10) into Eq.(6.6), one can compute P_{max} for $|W_3\rangle$ with $a_1 = a_2 = a$ and $s_3 = q$, whose final expression is simply

$$P_{max} = \frac{(1 - q^2)^2}{2 - 3q^2}. \tag{6.11}$$

Eq.(6.11) is consistent with Eq.(6.5) when $q^2 \leq 2a^2$. When $q = 0$, Eq.(6.11) gives $P_{max} = 1/2$ which corresponds to that of 2-qubit EPR state. When $q = 1/\sqrt{3}$, Eq.(6.11) gives $P_{max} = 4/9$, which is also consistent with the result of Ref.[93].

6.3 Four, five and six qubit W-type states: one-parametric cases

The method described in the previous section may enable us to compute P_{max} of four-qubit W-type states. For the case of arbitrary four-qubit systems P_{max} can be represented in a form

$$P_{max} = \frac{1}{8} \max_{|\vec{s}_1| = |\vec{s}_2| = |\vec{s}_3| = 1} \begin{bmatrix} 1 + \vec{s}_1 \cdot \vec{r}_1 + \vec{s}_2 \cdot \vec{r}_2 + \vec{s}_3 \cdot \vec{r}_3 + s_{1i}s_{2j}g_{ij}^{(3)} \\ +s_{1i}s_{3j}g_{ij}^{(2)} + s_{2i}s_{3j}g_{ij}^{(1)} + s_{1i}s_{2j}s_{3k}h_{ijk} \end{bmatrix}, \tag{6.12}$$

where

$$\begin{aligned}
&\vec{r}_1 = Tr[\rho^A \vec{\sigma}], \quad \vec{r}_2 = Tr[\rho^B \vec{\sigma}], \quad \vec{r}_3 = Tr[\rho^C \vec{\sigma}], \\
&g_{ij}^{(3)} = Tr[\rho^{AB} \sigma_i \otimes \sigma_j],\ g_{ij}^{(2)} = Tr[\rho^{AC} \sigma_i \otimes \sigma_j],\ g_{ij}^{(1)} = Tr[\rho^{BC} \sigma_i \otimes \sigma_j] \\
&h_{ijk} = Tr[\rho^{ABC} \sigma_i \otimes \sigma_j \otimes \sigma_k].
\end{aligned} \tag{6.13}$$

For the case of the generalized four-qubit W-state all vectors $\vec{r_k}$ are collinear, all matrices $g^{(k)}$ are diagonal and the vectors $\vec{r_k}$ are eigenvectors of the matrices $g^{(k)}$ as following:

$$\vec{r}_k = (0, 0, r_k), \quad g_{ij}^{(k)} = \begin{pmatrix} \omega_k & 0 & 0 \\ 0 & \omega_k & 0 \\ 0 & 0 & -\tilde{r}_k \end{pmatrix}, \quad k = 1, 2, 3. \tag{6.14}$$

In Eq.(6.14) we defined various quantities as following:

$$\begin{aligned}
&r_k = a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_k^2, \quad \omega_1 = 2a_2 a_3,\ \omega_2 = 2a_1 a_3,\ \omega_3 = 2a_1 a_2. \\
&\tilde{r}_1 = a_2^2 + a_3^2 - a_1^2 - a_4^2,\ \tilde{r}_2 = a_1^2 + a_3^2 - a_2^2 - a_4^2,\ \tilde{r}_3 = a_1^2 + a_2^2 - a_3^2 - a_4^2.
\end{aligned} \tag{6.15}$$

In addition, the non-vanishing components of h_{ijk} are

$$h_{113} = h_{223} = \omega_3 \qquad h_{131} = h_{232} = \omega_2 \qquad h_{311} = h_{322} = \omega_1 \qquad h_{333} = -r_4. \tag{6.16}$$

Due to the maximization in Eq.(6.12) the Bloch vectors should satisfy the following Lagrange multiplier equations:

$$\begin{aligned}
&r_{1i} + g_{ij}^{(3)} s_{2j} + g_{ij}^{(2)} s_{3j} + h_{ijk} s_{2j} s_{3k} = \Lambda_1 s_{1i} \\
&r_{2i} + g_{ji}^{(3)} s_{1j} + g_{ij}^{(1)} s_{3j} + h_{kij} s_{1k} s_{3j} = \Lambda_2 s_{2i} \\
&r_{3i} + g_{ji}^{(2)} s_{1j} + g_{ji}^{(1)} s_{2j} + h_{jki} s_{1j} s_{2k} = \Lambda_3 s_{3i}.
\end{aligned} \tag{6.17}$$

Now we put $s_{1y} = s_{2y} = s_{3y} = 0$ as before. After removing the Lagrange multiplier constants Λ_1, Λ_2 and Λ_3, one can show that Eq.(6.17) reduce to the following three equations:

$$\begin{aligned}
&s_{1x} \left[r_1 - \tilde{r}_3 s_{2z} - \tilde{r}_2 s_{3z} + \omega_1 s_{2x} s_{3x} - r_4 s_{2z} s_{3z} \right] = s_{1z} \left[\omega_2 s_{3x} (1 + s_{2z}) + \omega_3 s_{2x} (1 + s_{3z}) \right] \\
&s_{2x} \left[r_2 - \tilde{r}_3 s_{1z} - \tilde{r}_1 s_{3z} + \omega_2 s_{1x} s_{3x} - r_4 s_{1z} s_{3z} \right] = s_{2z} \left[\omega_1 s_{3x} (1 + s_{1z}) + \omega_3 s_{1x} (1 + s_{3z}) \right] \\
&s_{3x} \left[r_3 - \tilde{r}_1 s_{2z} - \tilde{r}_2 s_{1z} + \omega_3 s_{1x} s_{2x} - r_4 s_{1z} s_{2z} \right] = s_{3z} \left[\omega_2 s_{1x} (1 + s_{2z}) + \omega_1 s_{2x} (1 + s_{1z}) \right].
\end{aligned} \tag{6.18}$$

Eq.(6.18) implies that the Bloch vectors have the following symmetries:

$$\begin{aligned}
&\vec{s}_1(a_1,a_2,a_3,a_4) = \vec{s}_2(a_2,a_1,a_3,a_4) = \vec{s}_3(a_3,a_2,a_1,a_4) \\
&\vec{s}_1(a_1,a_2,a_3,a_4) = \vec{s}_1(a_1,a_3,a_2,a_4) \\
&\vec{s}_2(a_1,a_2,a_3,a_4) = \vec{s}_2(a_3,a_2,a_1,a_4) \\
&\vec{s}_3(a_1,a_2,a_3,a_4) = \vec{s}_3(a_2,a_1,a_3,a_4).
\end{aligned} \tag{6.19}$$

Therefore, one can compute all Bloch vectors if one of them is known. Using the symmetries (6.19), we can make single equation from Eq.(6.18) which is expressed in terms of s_{1z} only in a form

$$\frac{s_{1x}(a_1,a_2,a_3,a_4)}{s_{1z}(a_1,a_2,a_3,a_4)} = \frac{P(a_1,a_2,a_3,a_4)}{Q(a_1,a_2,a_3,a_4)} \tag{6.20}$$

where

$$\begin{aligned}
P(a_1,a_2,a_3,a_4) = &\ \omega_2\sqrt{1-s_{1z}^2(a_3,a_2,a_1,a_4)}\,[1+s_{1z}(a_2,a_1,a_3,a_4)] \\
&+\omega_3\sqrt{1-s_{1z}^2(a_2,a_1,a_3,a_4)}\,[1+s_{1z}(a_3,a_2,a_1,a_4)] \\
Q(a_1,a_2,a_3,a_4) = &\ r_1 - \tilde{r}_3 s_{1z}(a_2,a_1,a_3,a_4) - \tilde{r}_2 s_{1z}(a_3,a_2,a_1,a_4) \\
&+\omega_1\sqrt{1-s_{1z}^2(a_2,a_1,a_3,a_4)}\sqrt{1-s_{1z}^2(a_3,a_2,a_1,a_4)} \\
&-r_4 s_{1z}(a_2,a_1,a_3,a_4)s_{1z}(a_3,a_2,a_1,a_4).
\end{aligned}$$

Defining $a_1 = a_2 = a_3 \equiv a$ and $a_4 \equiv q$, one can solve Eq.(6.20) easily. The final expressions of solutions are

$$\begin{aligned}
s_{1z} = s_{2z} = s_{3z} &= \frac{1}{9a^2-q^2} \\
s_{1x} = s_{2x} = s_{3x} &= \sqrt{1-s_{1z}^2} = \frac{2\sqrt{6}a}{9a^2-q^2}\sqrt{3a^2-q^2}.
\end{aligned} \tag{6.21}$$

Inserting Eq.(6.21) into Eq.(6.12), one can compute P_{max} for $|W_4\rangle$ with $a_1 = a_2 = a_3 \equiv a$ and $a_4 \equiv q$ whose final expression is

$$P_{max} = \frac{2^2(1-q^2)^3}{(3-4q^2)^2}. \tag{6.22}$$

Eq.(6.21) implies that P_{max} in Eq.(6.22) is valid when $q^2 \le 3a^2$. When $q = 0$, P_{max} becomes $4/9$ as expected. When $q = 1/2$, P_{max} becomes $27/64$, which is in agreement with the result of Ref.[93].

One can repeat the calculation for $|W_5\rangle$ with $a_1 = a_2 = a_3 = a_4 \equiv a$ and $a_5 = q$. Then the final expression of P_{max} becomes

$$P_{max} = \frac{3^3(1-q^2)^4}{(4-5q^2)^3}. \tag{6.23}$$

When $q = 0$, P_{max} reduces to $27/64$ as expected. When $q = 1/\sqrt{5}$, P_{max} reduces to $(4/5)^4$. By the same way P_{max} for $|W_6\rangle$ can be written as

$$P_{max} = \frac{4^4(1-q^2)^5}{(5-6q^2)^4}. \tag{6.24}$$

Fig. 1 is a plot of q-dependence of P_{max} for $|W_4\rangle$, $|W_5\rangle$ and $|W_6\rangle$. The black dots are numerical data computed by the numerical technique exploited in Ref.[93]. The red solid and red dotted lines are Eq.(6.22), Eq.(6.23) and Eq.(6.24) when $q \leq 1/\sqrt{2}$ and $q \geq 1/\sqrt{2}$ respectively. As expected the numerical data are in perfect agreement with Eq.(6.22), Eq.(6.23) and Eq.(6.24) in the applicable domain, *i.e.* $q^2 \leq (n-1)a^2$ for $|W_n\rangle$. Outside the applicable domain ($q^2 \geq 1/\sqrt{2}$) the numerical data are in disagreement with these equations.

6.4 General multi-qubit W-type states: one-parametric cases

From Eq.(6.11), (6.22), (6.23) and (6.24) one can guess that P_{max} for W_n is ($a_1 = \cdots = a_{n-1} \equiv a$, $a_n \equiv q$)

$$P_{max}(n,q) = (1-q^2)^{n-1} \left(\frac{n-2}{(n-1) - nq^2} \right)^{n-2}. \tag{6.25}$$

Using this result, one can straightforwardly construct the nearest product state to $|W_n\rangle$. After some algebra, when $q^2 \leq (n-1)a^2$, one can show that the analytic expression of the nearest product state is $|q_1\rangle \otimes |q_2\rangle \otimes \cdots \otimes |q_n\rangle$, where

$$\begin{aligned} &|q_1\rangle = \cdots = |q_{n-1}\rangle = \\ &\qquad \frac{1}{\sqrt{(n-1)^2a^2 - q^2}} \left[\sqrt{(n-1)(n-2)}a|0\rangle + \sqrt{(n-1)a^2 - q^2}e^{i\varphi}|1\rangle \right] \\ &|q_n\rangle = \frac{1}{\sqrt{(n-1)^2a^2 - q^2}} \left[\sqrt{(n-1)^2a^2 - (n-1)q^2}|0\rangle + \sqrt{n-2}qe^{i\varphi}|1\rangle \right] \end{aligned} \tag{6.26}$$

and φ is an arbitrary phase factor. When $q^2 \geq (n-1)a^2$, the nearest product state, of course, becomes $|0\cdots01\rangle$.

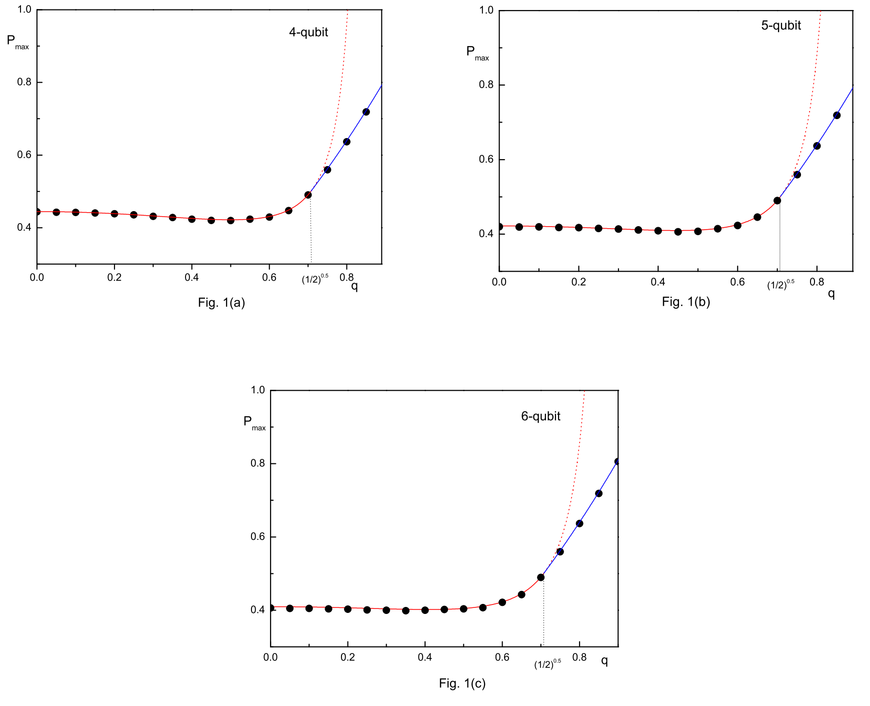

Figure 6.1: Plot of q-dependence of P_{max} for 4-qubit (Fig. 1(a)), 5-qubit (Fig. 1(b)), and 6-qubit (Fig. 1(c)). The black dots are numerical data of P_{max}. The red solid lines are result of Eq.(6.25) in the applicable domain, $0 \leq q \leq 1/\sqrt{2}$. The red dotted lines are result of Eq.(6.25) outside the applicable domain. The blue solid lines are plot of $\max(a^2, q^2) = q^2$ outside the applicable domain. This figures strongly suggest that P_{max} for $|W_n\rangle$ is Eq.(6.25) when $q \leq 1/\sqrt{2}$ and $\max(a^2, q^2) = q^2$ when $q \geq 1/\sqrt{2}$.

Now, we present a simple proof for both equations (6.25) and (6.26). It is easy to check

$$\langle q_1 q_2 \cdots q_{n-1} | W_n \rangle = e^{-i\varphi} \sqrt{P_{max}} |q_n\rangle, \quad \langle q_2 q_3 \cdots q_{n-1} q_n | W_n \rangle = e^{-i\varphi} \sqrt{P_{max}} |q_1\rangle. \tag{6.27}$$

The second equation in (6.27) is invariant under the permutations $(q_1 \leftrightarrow q_j, j = 2, 3, \cdots n-1)$. Thus, the product state satisfies the stationarity equations of Ref.[43] and consequently, is the nearest separable state. Accordingly, $\sqrt{P_{max}}$ is the injective tensor norm of $|W_n\rangle$.

When $q = 0$ and $q = 1/\sqrt{n}$, P_{max} reduces to $(1 - 1/(n-1))^{n-2}$ and $(1 - 1/n)^{n-1}$ respectively. Thus, Eq.(6.25) is perfectly in agreement with the result of Ref.[93]. Another interesting point in Eq.(6.25) is that P_{max} becomes $1/2$ regardless of n when $q = 1/\sqrt{2}$, the boundary of the applicable domain. This makes us conjecture that outside the applicable domain P_{max} becomes $\max(a^2, q^2) = q^2$ like 3-qubit case. The blue solid lines in Fig. 1 are plot of q^2 at the domain $q \geq 1/\sqrt{2}$. As we conjecture, the blue lines are perfectly in agreement of numerical data.

Another consequence of Eq.(6.25) is the entanglement witness $\hat{W}_n$ for an one-parametric W-type state. Its construction is straightforward as following form:

$$\hat{W}_n = P_{max}(n, q)\mathbb{1} - |W_n(q)\rangle\langle W_n(q)|, \tag{6.28}$$

where $\mathbb{1}$ is a unit matrix. Obviously one can show

$$Tr\left(\hat{W}_n |W_n(q)\rangle\langle W_n(q)|\right) < 0, \quad Tr\left(\hat{W}_n \rho_0\right) \geq 0, \tag{6.29}$$

where ρ_0 is any separable state. Thus, $\hat{W}_n$ is an entanglement witness and allows an experimental detection of the multipartite entanglement.

6.5 Four-qubit W state: two-parametric cases

In this section we will compute P_{max} for the two-parametric $|W_4\rangle$ given by

$$|W_4\rangle = a|1000\rangle + b|0100\rangle + q|0010\rangle + q|0001\rangle. \tag{6.30}$$

It seems to be difficult to apply the Lagrange multiplier method directly due to their non-trivial nonlinearity. Thus, we will adopt the usual maximization method.

The maximum overlap probability P_{max} is

$$P_{max} = \max_{|q_1\rangle|q_2\rangle|q\rangle} |\langle q_1|\langle q_2|\langle q|\langle q \mid W_4\rangle|^2. \tag{6.31}$$

Now we define the 1-qubit states as $|q_1\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, $|q_2\rangle = \beta_0|0\rangle + \beta_1|1\rangle$ and $|q\rangle = \gamma_0|0\rangle + \gamma_1|1\rangle$. For simplicity, we are assuming that all coefficients are real and positive. Then, P_{max} becomes

$$P_{max} = \max_{\alpha_0,\beta_0,\gamma_0} \gamma_0^2 \left(a\beta_0\gamma_0\sqrt{1-\alpha_0^2} + b\alpha_0\gamma_0\sqrt{1-\beta_0^2} + 2q\alpha_0\beta_0\sqrt{1-\gamma_0^2}\right)^2. \tag{6.32}$$

Since the maximum value is determined at extremum point, it is useful if the extremum conditions are derived. This is achieved by differentiating Eq.(6.32), which leads to

$$\begin{aligned}
&b\gamma_0\sqrt{1-\beta_0^2}+2q\beta_0\sqrt{1-\gamma_0^2}=a\beta_0\gamma_0\frac{\alpha_0}{\sqrt{1-\alpha_0^2}}\\
&a\gamma_0\sqrt{1-\alpha_0^2}+2q\alpha_0\sqrt{1-\gamma_0^2}=b\alpha_0\gamma_0\frac{\beta_0}{\sqrt{1-\beta_0^2}}\\
&a\beta_0\gamma_0\sqrt{1-\alpha_0^2}+b\alpha_0\gamma_0\sqrt{1-\beta_0^2}+q\alpha_0\beta_0\sqrt{1-\gamma_0^2}=q\alpha_0\beta_0\frac{\gamma_0^2}{\sqrt{1-\gamma_0^2}}.
\end{aligned} \tag{6.33}$$

One can solve the equations by separating α_0 from β_0, γ_0, i.e.,

$$\begin{aligned}
\frac{\alpha_0}{\sqrt{1-\alpha_0^2}}&=\frac{b}{a}\frac{\sqrt{1-\beta_0^2}}{\beta_0}+\frac{2q}{a}\frac{\sqrt{1-\gamma_0^2}}{\gamma_0}\\
\frac{\sqrt{1-\alpha_0^2}}{\alpha_0}&=\frac{b}{a}\frac{\beta_0}{\sqrt{1-\beta_0^2}}-\frac{2q}{a}\frac{\sqrt{1-\gamma_0^2}}{\gamma_0}\\
\frac{\sqrt{1-\alpha_0^2}}{\alpha_0}&=\frac{q}{a}\frac{\gamma_0}{\sqrt{1-\gamma_0^2}}-\frac{q}{a}\frac{\sqrt{1-\gamma_0^2}}{\gamma_0}-\frac{b}{a}\frac{\sqrt{1-\beta_0^2}}{\beta_0}
\end{aligned} \tag{6.34}$$

and one can get the solutions for β_0 and γ_0 as follows:

$$\begin{aligned}
\beta_0^2&=\frac{3}{2}-\frac{4q^2-a^2+b^2}{4q^2}\gamma_0^2\\
\gamma_0^2&=\frac{4q^2(4q^2-a^2-b^2)-2q^2\sqrt{(4q^2-a^2-b^2)^2+12a^2b^2}}{(4q^2+b^2-a^2)^2-16q^2b^2}.
\end{aligned} \tag{6.35}$$

The solution for α_0 is obtained by separating β_0:

$$\alpha_0^2=\frac{3}{2}-\frac{4q^2+a^2-b^2}{4q^2}\gamma_0^2. \tag{6.36}$$

Inserting these extremum solution in P_{max} and rationalizing denominator, one gets

$$P_{max}=\frac{2q^4\left[(4q^2-a^2-b^2)\left\{(4q^2-a^2-b^2)^2-36a^2b^2\right\}+\left\{(4q^2-a^2-b^2)^2+12a^2b^2\right\}^{\frac{3}{2}}\right]}{\left\{(4q^2-a^2-b^2)^2-4a^2b^2\right\}^2}. \tag{6.37}$$

Of course, Eq.(6.37) is valid when $\alpha^2\leq\beta^2+\gamma^2+\delta^2$, where $\{\alpha,\beta,\gamma,\delta\}$ is $\{a,b,q,q\}$ with decreasing order. When $\alpha^2\geq\beta^2+\gamma^2+\delta^2$, P_{max} will be $\alpha^2=\max(a^2,b^2)$.

The dependence of the maximal overlap on state parameters is shown in Fig.2. The behavior of P_{max} in different limits is explained in the next section.

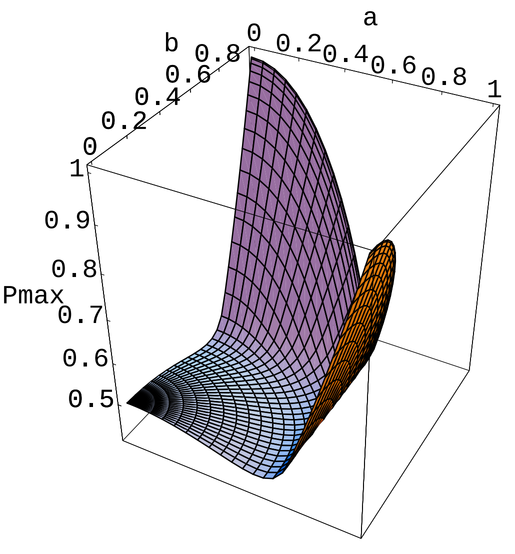

Figure 6.2: The maximal overlap P_{max} vs. the parameters a and b for the 4-qubit state. The green and blue areas are highly entangled regions and the maximal overlap is given by Eq.(6.37). The violet(dark orange) area is a slightly entangled region and the maximal overlap is $\max(a^2, b^2)$. It is minimal ($P_{max} = 27/64$) at $a = b = 1/2$ which is the W-state and maximal ($P_{max} = 1$) either at $a = 1, b = 0$ or at $a = 0, b = 1$ which are product states.

6.6 Special four-qubit W-type states

In this section we consider some special 4-qubit states.

The first one is $a = 0$ limit. Since $|W_4\rangle = |0\rangle \otimes (b|100\rangle + q|010\rangle + q|001\rangle)$ in this limit, one can compute P_{max} using Eq.(6.5). In this limit Eq.(6.37) gives

$$P_{max} = \frac{4q^4}{4q^2 - b^2} \qquad (b^2 \leq 2q^2). \tag{6.38}$$

One can show easily that this is perfectly in agreement with Eq.(6.5).

The second special case is $a = q$ limit. In this limit Eq.(6.37) gives

$$P_{max} = \frac{4(1 - b^2)^3}{(3 - 4b^2)^2} \qquad (b^2 \leq 3q^2) \tag{6.39}$$

which is also consistent with Eq.(6.22).

The last special case is $2q = a + b$ limit. Although both denominator and numerator in Eq.(6.37) vanish, their ratio has a finite limit and P_{max} takes correct values in the applicable domain. The applicable domain is defined by the two restrictions $\alpha^2 \leq \beta^2 + \gamma^2 + \delta^2$ and $2q = a + b$. These restrictions together with the normalization condition impose upper and lower bounds for the parameters a and b

$$\min(a, b) \geq \frac{\sqrt{2}}{6}, \quad \max(a, b) \leq \frac{\sqrt{2}}{2}. \tag{6.40}$$

The maximum overlap probability P_{max} is

$$P_{max} = \frac{27}{256} \frac{(a + b)^4}{ab}. \tag{6.41}$$

The limit $a = b = q = 1/2$ again yields P_{max}=27/64. Another interesting limit is the case when $b(a)$ is minimal and $a(b)$ is maximal. This limit is reached at $a = 3b(b = 3a)$. Then Eq.(6.41) yields $P_{max} = 1/2 = \alpha^2$. These states are first type shared states[67] and allow perfect teleportation and superdense coding scenario.

6.7 discussion

We have calculated the maximal overlap of one- and two-parametric W-type states and found their nearest separable states. However, in some sub-region of the parameter space one can find the nearest states and corresponding maximal overlaps for generic W-type states. In fact, the square of any coefficient in Eq.(6.2) is a maximal overlap in some region of state parameters. It is easy to check that the product state $|0_1...0_{k-1}1_k0_{k+1}...0_n\rangle$ is a solution of stationarity equation with entanglement eigenvalue $\sqrt{P_{max}} = a_k$. From previous results one can guess that this solution gives a true maximum of the overlap if

$$a_k^2 \geq a_1^2 + a_2^2 + \cdots + a_{k-1}^2 + a_{k+1}^2 + \cdots + a_n^2 = 1 - a_k^2. \tag{6.42}$$

Then the maximal overlap in the slightly entangled region can be written readily in the form

$$P_{max} = \max(a_1^2, a_2^2, \cdots, a_n^2) \quad \text{if} \quad \max(a_1^2, a_2^2, \cdots, a_n^2) \geq \frac{1}{2}. \tag{6.43}$$

This formula has the following simple interpretation. Equation (6.42) means that the state is already written in the Schmidt normal form and the maximal overlap takes the value of the largest coefficient [131].

Now the question at issue is what is happening if $a_k^2 < 1/2,\ k = 1, 2, \cdots, n$. From these inequalities it follows that

$$\frac{1}{2}\left(a_1 + a_2 + \cdots + a_n\right) > \max(a_1, a_2, \cdots, a_n). \tag{6.44}$$

From any set of such coefficients one can form polygons(polyhedrons). This fact is an indirect evidence that P_{max} has a geometrical meaning. Unfortunately, there is an obstacle to the goal achievement. The problem is that we have not the answer for generic states. For example, it is difficult to conclude from Eq.(6.11) that the expression is the circumradius of a triangle in a particular limit. In general, one can form many polygons, either convex or crossed, from the set $a_1, a_2, ..., a_n$. Each of them generates its own geometric quantities that can be treated as the maximal overlap. This happens because stationarity equations have many solutions in highly entangled region. And all of these solutions yield the same expression in particular cases. For example, in Ref.[67] it was shown that all convex and crossed quadrangles are contracted to the same triangle in particular limits. In conclusion, in order to find a true geometric interpretation one has to derive P_{max} for generic states.

Another(and probably promising) way to get the desired interpretation is the following. Since the surface $(a_1^2 - 1/2)(a_2^2 - 1/2)\cdots(a_n^2 - 1/2) = 0$ separates highly and slightly entangled regions, one may ask what is happening on this surface. That is, we are considering polygons whose sides satisfy the equality $a_k^2 = a_1^2 + a_2^2 + \cdots + a_{k-1}^2 + a_{k+1}^2 + \cdots + a_n^2$ for any k. For $n = 3$ we perfectly know that corresponding polygons are right triangles and the center of a circumcircle lies on the largest side of a right triangle. Then, we can conclude that if the center of the circumcircle is inside the triangle, then the maximal overlap is the circumradius and otherwise is the largest coefficient. However, for $n \geq 4$ we do not know what are the polygons for which the square of the largest side is the sum of squares of the remaining coefficients. If one understands the geometric meaning of this relation, then one finds a clue. And this clue may enable us to find P_{max} for generic W-type states. These type of analytic expressions can have practical application in QICC and may shed new light on multipartite entanglement.

All above-mentioned problems owe their origin to the fact that the injective tensor norm is related to the Cayley's Hyperdeterminant [133]. It is well-known that this hyperdeterminant has a geometrical interpretation for $n = 3$ and no such interpretation is known for $n \geq 4$ so far. We hope to keep on studying this issue in the future.

Part II

Entanglement for mixed states

Chapter 7

Three-tangle for rank-three mixed states: Mixture of Greenberger-Horne-Zeilinger, W, and flipped-W states

Entanglement is a genuine physical resource for the quantum information theories[15]. It is at the heart of the recent much activities on the research of quantum computer. Although many new results have been derived recently for the entanglement of pure states[43, 64, 66, 67, 69], entanglement for mixed states is not much understood so far compared to the pure states. Since, however, the effect of environment generally changes the pure state into the mixed state, it is highly important to investigate the entanglement of the mixed states.

Entanglement for the bipartite mixed states, called concurrence, was studied by Hill and Wootters in Ref.[38] when the density matrix of the state has two or more zero-eigenvalue. Subsequently, Wootters extended the result of Ref.[38] to the arbitrary bipartite mixed states[39] by making use of the time reversal operator of the spin-1/2 particle appropriately. In addition, the concurrence was used to derive the purely tripartite entanglement called residual entanglement or three-tangle[71]. For three-qubit pure state $|\psi\rangle = \sum_{i,j,k=0}^{1} a_{ijk}|ijk\rangle$, the three-tangle τ_3 becomes[71]

$$\tau_3 = 4|d_1 - 2d_2 + 4d_3|, \tag{7.1}$$

where

$$\begin{aligned} d_1 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\ d_2 &= a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} + a_{000}a_{111}a_{110}a_{001} \\ &\quad + a_{011}a_{100}a_{101}a_{010} + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001} \\ d_3 &= a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}. \end{aligned} \tag{7.2}$$

The three-tangle is polynomial invariant under the local $SL(2, \mathbb{C})$ transformation[74, 75] and exactly coincides with the modulus of a Cayley's hyperdeterminant[72, 73]. For the mixed three-qubit state ρ the three-tangle is defined by making use of the convex roof construction[36, 137] as

$$\tau_3(\rho) = \min \sum_i p_i \tau_3(\rho_i), \tag{7.3}$$

where minimum is taken over all possible ensembles of pure states. The ensemble corresponding to the minimum of τ_3 is called optimal decomposition.

Although the definition of three-tangle for the mixed states is simple as shown in Eq.(7.3), it is highly difficult to compute it. This is mainly due to the fact that the construction of the optimal decomposition for the arbitrary state is a formidable task. Even for the most simple case of rank-two state still we do not know how to construct the optimal decomposition except very rare cases.

Recently, Ref.[76] has shown how to construct the optimal decomposition for the rank-2 mixture of Greenberger-Horne-Zeilinger(GHZ) and W states:

$$\rho(p) = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W|, \tag{7.4}$$

where

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \qquad |W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right). \tag{7.5}$$

The optimal decomposition for $\rho(p)$ was constructed with use of the fact that $\tau_3(|GHZ\rangle) = 1$, $\tau_3(|W\rangle) = 0$ and $\langle GHZ|W\rangle = 0$. Once the optimal decompositions are constructed, it is easy to compute the three-tangle. For $\rho(p)$ the three-tangle has three-different expressions depending on the range of p as following:

$$\tau_3(\rho(p)) = \begin{cases} 0 & \text{for } 0 \leq p \leq p_0 \\ g_I(p) & \text{for } p_0 \leq p \leq p_1 \\ g_{II}(p) & \text{for } p_1 \leq p \leq 1 \end{cases} \tag{7.6}$$

where

$$g_I(p) = p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1-p)^3} \qquad g_{II}(p) = 1-(1-p)\left(\frac{3}{2}+\frac{1}{18}\sqrt{465}\right)$$

$$p_0 = \frac{4\sqrt[3]{2}}{3+4\sqrt[3]{2}} \sim 0.6269 \qquad p_1 = \frac{1}{2}+\frac{3}{310}\sqrt{465} \sim 0.7087. \tag{7.7}$$

More recently, this result was extended to the rank-2 mixture of generalized GHZ and generalized W states in Ref.[78].

The purpose of this chapter is to extend Ref.[76] to the case of rank-3 mixed states. In this chapter we would like to analyze the optimal decompositions for the mixture of GHZ, W and flipped W states as

$$\rho(p,q) = p|GHZ\rangle\langle GHZ| + q|W\rangle\langle W| + (1-p-q)|\tilde{W}\rangle\langle\tilde{W}|, \tag{7.8}$$

where

$$|\tilde{W}\rangle = \frac{1}{\sqrt{3}}\left(|110\rangle + |101\rangle + |011\rangle\right). \tag{7.9}$$

For simplicity, we will define q as

$$q = \frac{1-p}{n}, \tag{7.10}$$

where n is positive integer. Before we go further, it is worthwhile noting that $\rho(p,q) = \rho(p)$ when $n=1$ and therefore, Eq.(7.6) is the three-tangle in this case. When $n=\infty$, $\rho(p,q)$ can be constructed from $\rho(p)$ by local-unitary (LU) transformation $\sigma_x\otimes\sigma_x\otimes\sigma_x$. Since the three-tangle is LU-invariant quantity, the three-tangle of $\rho(p,q)$ with $n=\infty$ is again Eq.(7.6).

Now we start with three-qubit pure state

$$|Z(p,q,\varphi_1,\varphi_2)\rangle = \sqrt{p}|GHZ\rangle - e^{i\varphi_1}\sqrt{q}|W\rangle - e^{i\varphi_2}\sqrt{1-p-q}|\tilde{W}\rangle \tag{7.11}$$

whose three-tangle is

$$\tau_3(p,q,\varphi_1,\varphi_2) = \left| \begin{array}{l} p^2 - 4p\sqrt{q(1-p-q)}e^{i(\varphi_1+\varphi_2)} - \frac{4}{3}q(1-p-q)e^{2i(\varphi_1+\varphi_2)} \\ -\frac{8\sqrt{6}}{9}\sqrt{pq^3}e^{3i\varphi_1} - \frac{8\sqrt{6}}{9}\sqrt{p(1-p-q)^3}e^{3i\varphi_2} \end{array} \right|. \tag{7.12}$$

The state $|Z(p,q,\varphi_1,\varphi_2)\rangle$ has several interesting properties. Firstly, the mixed state $\rho(p,q)$ in Eq.(7.8) can be expressed in terms of $|Z(p,q,\varphi_1,\varphi_2)\rangle$ as following:

$$\rho(p,q) = \frac{1}{3}\left[\begin{array}{l} |Z(p,q,0,0)\rangle\langle Z(p,q,0,0)| + |Z\left(p,q,\frac{2\pi}{3},\frac{4\pi}{3}\right)\rangle\langle Z\left(p,q,\frac{2\pi}{3},\frac{4\pi}{3}\right)| \\ +|Z\left(p,q,\frac{4\pi}{3},\frac{2\pi}{3}\right)\rangle\langle Z\left(p,q,\frac{4\pi}{3},\frac{2\pi}{3}\right)| \end{array} \right] \tag{7.13}$$

Secondly, $|Z(p,q,0,0)\rangle$, $|Z\left(p,q,\frac{2\pi}{3},\frac{4\pi}{3}\right)\rangle$ and $|Z\left(p,q,\frac{4\pi}{3},\frac{2\pi}{3}\right)\rangle$ have same three-tangle as shown from Eq.(7.12) directly. Thirdly, the numerical calculation shows that the p-dependence of $\tau_3(p,(1-p)/n,\varphi_1,\varphi_2)$ has many zeros depending on φ_1 and φ_2, but the largest zero $p = p_0$ arises when $\varphi_1 = \varphi_2 = 0$ regardless of n. It can be proven rigorously with use of the implicit function theorem. The n-dependence of p_0 is given in Table I. Table I indicates that when n increases from $n = 2$, p_0 approaches to $4\sqrt[3]{2}/(3+4\sqrt[3]{2}) \sim 0.6269$ from $3/4 = 0.75$. This is because of the fact that the three-tangle for $\rho(p,q)$ should be Eq.(7.6) in the $n \to \infty$ limit.

When $p \leq p_0$, one can construct the optimal decomposition by making use of Eq.(7.13) as following:

$$\begin{aligned}\rho\left(p,\frac{1-p}{n}\right) = \frac{p}{3p_0}\Bigg[&|Z\left(p_0,\frac{1-p_0}{n},0,0\right)\rangle\langle Z\left(p_0,\frac{1-p_0}{n},0,0\right)| \\ &+|Z\left(p_0,\frac{1-p_0}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)\rangle\langle Z\left(p_0,\frac{1-p_0}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)| \\ &+|Z\left(p_0,\frac{1-p_0}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)\rangle\langle Z\left(p_0,\frac{1-p_0}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)|\Bigg] \\ &+\frac{p_0-p}{np_0}|W\rangle\langle W| + \frac{(n-1)(p_0-p)}{np_0}|\tilde{W}\rangle\langle\tilde{W}|.\end{aligned} \tag{7.14}$$

Thus, we have vanishing three-tangle in this region:

$$\tau_3\left[\rho\left(p,\frac{1-p}{n}\right)\right] = 0 \qquad \text{for } p \leq p_0. \tag{7.15}$$

Now, we consider $p_0 \leq p \leq 1$ region. When $p = p_0$, Eq.(7.14) implies that the optimal decomposition consists of three pure states $|Z\left(p_0,\frac{1-p_0}{n},0,0\right)\rangle$, $|Z\left(p_0,\frac{1-p_0}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)\rangle$, and $|Z\left(p_0,\frac{1-p_0}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)\rangle$ with same probability. This fact together with Eq.(7.13) strongly suggests that the optimal decomposition at $p_0 \leq p$ is described by Eq.(7.13). As will be shown below, however, this is not true in the full range of $p_0 \leq p \leq 1$.

The optimal decomposition (7.13) gives the three-tangle to $\rho(p,q)$ in a form

$$\alpha_I(p) = p^2 - \frac{4\sqrt{n-1}}{n}p(1-p) - \frac{4(n-1)}{3n^2}(1-p)^2 - \frac{8\sqrt{6n}\left[1+(n-1)^{3/2}\right]}{9n^2}\sqrt{p(1-p)^3}. \tag{7.16}$$

Since the three-tangle for mixed state is defined as a convex roof, $\alpha_I(p)$ should be convex function if it is a correct three-tangle in the range of $p_0 \leq p \leq 1$. In order to check this we compute $d^2\alpha_I/dp^2$, which is

$$\frac{d^2\alpha_I(p)}{dp^2} = \frac{2}{9n^2}\left[\left\{9n^2 + 36n\sqrt{n-1} - 12(n-1)\right\} - \sqrt{6n}\left\{1+(n-1)^{3/2}\right\}\frac{8p^2-4p-1}{\sqrt{p^3(1-p)}}\right]. \tag{7.17}$$

n	1	2	3	10	100	1000
p_0	0.6269	0.75	0.7452	0.712	0.6604	0.6382
p_1	0.7087	0.9330	0.9250	0.8667	0.7710	0.7298
p_*	0.8257	0.9618	0.9572	0.9230	0.8650	0.8391

Table 7.1: The n-dependence of p_0, p_1 and p_*.

Using Eq.(7.17) one can show that $d^2\alpha_I(p)/dp^2 \leq 0$ when $p_* \leq p \leq 1$. The n-dependence of p_* is given in Table I. Thus, we need to convexify $\alpha_I(p)$ in the region $p_1 \leq p \leq 1$, where $p_1 \leq p_*$. The constant p_1 will be determined shortly.

For large p region one can construct the optimal decomposition as following:

$$\begin{aligned}
\rho(p,q) &= p|GHZ\rangle\langle GHZ| + \frac{1-p}{n}|W\rangle\langle W| + \frac{(n-1)(1-p)}{n}|\tilde{W}\rangle\langle\tilde{W}| \qquad (7.18)\\
&= p|GHZ\rangle\langle GHZ| + \frac{1-p}{1-p_1}\Bigg[-p_1|GHZ\rangle\langle GHZ| + p_1|GHZ\rangle\langle GHZ| \\
&\qquad\qquad + \frac{1-p_1}{n}|W\rangle\langle W| + \frac{(n-1)(1-p_1)}{n}|\tilde{W}\rangle\langle\tilde{W}|\Bigg] \\
&= \frac{p-p_1}{1-p_1}|GHZ\rangle\langle GHZ| \\
&\quad + \frac{1-p}{3(1-p_1)}\Bigg[|Z\left(p_1, \frac{1-p_1}{n}, 0, 0\right)\rangle\langle Z\left(p_1, \frac{1-p_1}{n}, 0, 0\right)| \\
&\qquad + |Z\left(p_1, \frac{1-p_1}{n}, \frac{2\pi}{3}, \frac{4\pi}{3}\right)\rangle\langle Z\left(p_1, \frac{1-p_1}{n}, \frac{2\pi}{3}, \frac{4\pi}{3}\right)| \\
&\qquad + |Z\left(p_1, \frac{1-p_1}{n}, \frac{4\pi}{3}, \frac{2\pi}{3}\right)\rangle\langle Z\left(p_1, \frac{1-p_1}{n}, \frac{4\pi}{3}, \frac{2\pi}{3}\right)|\Bigg]
\end{aligned}$$

which gives the three-tangle in a form

$$\alpha_{II}(p) = \frac{p-p_1}{1-p_1} + \frac{1-p}{1-p_1}\alpha_I(p_1). \qquad (7.19)$$

Note that $d^2\alpha_{II}(p)/dp^2 = 0$. Thus, $\alpha_{II}(p)$ does not violate the convex constraint of the three-tangle in the large p region. The parameter p_1 is determined by minimizing $\alpha_{II}(p)$, *i.e.* $\partial\alpha_{II}/\partial p_1 = 0$, which gives

$$\frac{4\sqrt{6n}\left[1+(n-1)^{3/2}\right]}{9n^2}\frac{2p_1-1}{\sqrt{p_1(1-p_1)}} = 1 + \frac{4\sqrt{n-1}}{n} - \frac{4(n-1)}{3n^2}. \qquad (7.20)$$

The n-dependence of p_1 is given in Table I. As expected p_1 is between p_0 and

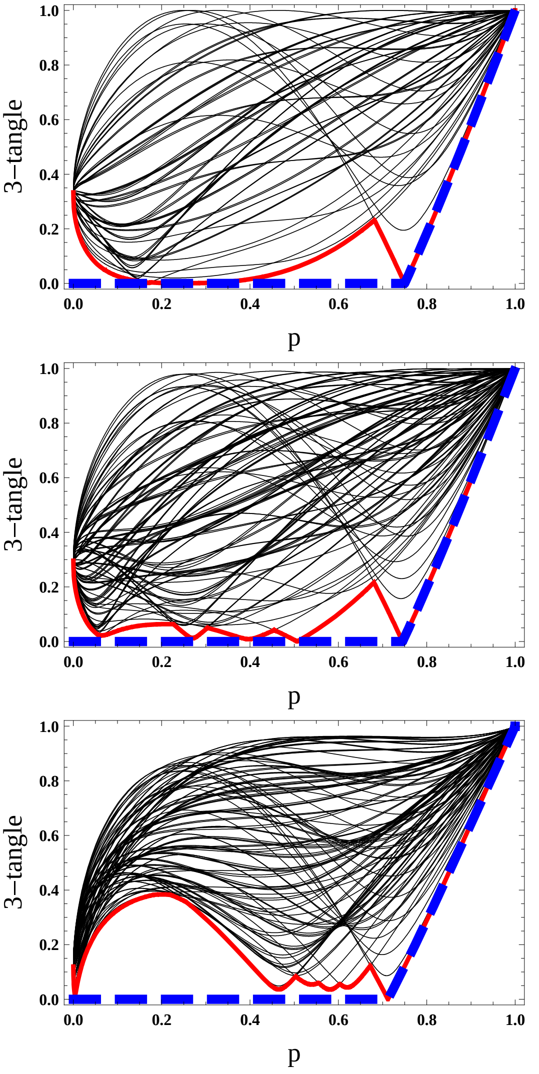

Figure 7.1: The plot of p-dependence of the Eq.(7.12) for various φ_1 and φ_2. We have chosen φ_1 and φ_2 from 0 to 2π as an interval 0.3. The three figures correspond to $n = 2$ (Fig. 7.1a), $n = 3$ (Fig. 7.1b) and $n = 10$ (Fig. 7.1c) respectively. The minimum curve is plotted as a thick solid line in each figure. These figures indicate that the three-tangle in Eq.(7.21) (plotted as dashed line in each figure) is a convex hull of the thick solid line.

p_*. When n increases from $n = 2$, p_1 decreases from $(2 + \sqrt{3})/4 \sim 0.933$ to $1/2 + 3\sqrt{465}/310 \sim 0.709$.

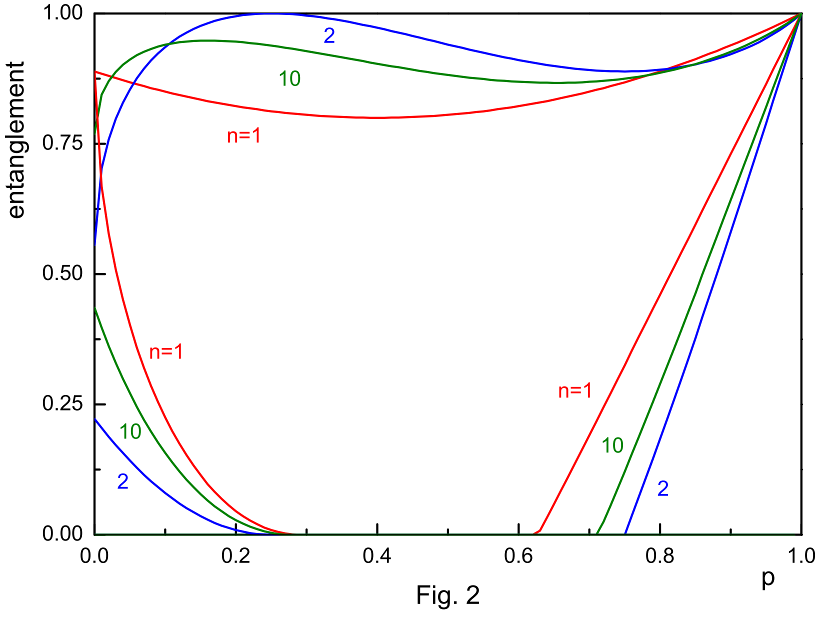

Figure 7.2: The p-dependence of one-tangle (upper solidlines), sum of squared concurrences (left solid lines) and three-tangle (right solid lines) for $n = 1$, 2 and 10. This figure clearly indicates that not only CKW inequality (7.25) but also (7.28) hold for all integer n.

In summary, the three-tangle for $\rho(p, q)$ is

$$\tau_3(\rho(p,q)) = \begin{cases} 0 & \text{for } 0 \leq p \leq p_0 \\ \alpha_I(p) & \text{for } p_0 \leq p \leq p_1 \\ \alpha_{II}(p) & \text{for } p_1 \leq p \leq 1 \end{cases} \tag{7.21}$$

and the corresponding optimal decompositions are (7.14), (7.13), and (7.18) respectively. In order to show that Eq.(7.21) is genuine optimal, we plotted the p-dependence of the three-tangles (7.12) for various φ_1 and φ_2 when $n = 2$ (Fig. 7.1a),

$n = 3$ (Fig. 7.1b) and $n = 10$ (Fig. 7.1c). These curves have been referred as the characteristic curves[79]. As Ref.[79] indicated, the three-tangle is a convex hull of the minimum of the characteristic curves (thick solid lines in the figure). Fig. 7.1 indicates that the three-tangles (7.21) plotted as dashed lines are the convex characteristic curves, which implies that Eq.(7.21) is really optimal.

The above analysis can be applied to provide an analytical technique which decides whether or not an arbitrary rank-3 state has vanishing three-tangle. First we correspond our states to the qutrit states with

$$|GHZ\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |W\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\tilde{W}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.22}$$

It is well-known[138] that the density matrix of the arbitrary qutrit state can be represented by $\rho = (1/3)(I + \sqrt{3}\vec{n} \cdot \vec{\lambda})$, where $\vec{n}$ is 8-dimensional unit vector and λ_i $(i = 1, \cdots, 8)$ are Gell-Mann matrices. Thus the points on the S^8 correspond to pure qutrit states while the interior points denote the mixed states[1]. Then, one can show straightforwardly that the pure states with vanishing three-tangle correspond to

$$\begin{aligned}
&|W\rangle \rightarrow \left(0, 0, -\frac{\sqrt{3}}{2}, 0, 0, 0, 0, \frac{1}{2}\right) && (7.23)\\
&|\tilde{W}\rangle \rightarrow (0, 0, 0, 0, 0, 0, 0, -1) \\
&|Z\left(p_0, \frac{1-p_0}{n}, 0, 0\right)\rangle \rightarrow \left(-\sqrt{3}\xi_1, 0, \eta_1, -\sqrt{3}\xi_2, 0, \sqrt{3}\xi_3, 0, \eta_2\right) \\
&|Z\left(p_0, \frac{1-p_0}{n}, \frac{2\pi}{3}, \frac{4\pi}{3}\right)\rangle \rightarrow \left(\frac{\sqrt{3}}{2}\xi_1, -\frac{3}{2}\xi_1, \eta_1, \frac{\sqrt{3}}{2}\xi_2, \frac{3}{2}\xi_2, -\frac{\sqrt{3}}{2}\xi_3, \frac{3}{2}\xi_3, \eta_2\right) \\
&|Z\left(p_0, \frac{1-p_0}{n}, \frac{4\pi}{3}, \frac{2\pi}{3}\right)\rangle \rightarrow \left(\frac{\sqrt{3}}{2}\xi_1, \frac{3}{2}\xi_1, \eta_1, \frac{\sqrt{3}}{2}\xi_2, -\frac{3}{2}\xi_2, -\frac{\sqrt{3}}{2}\xi_3, -\frac{3}{2}\xi_3, \eta_2\right),
\end{aligned}$$

where $\xi_1 = \sqrt{p_0(1-p_0)/n}$, $\xi_2 = \sqrt{n-1}\xi_1$, $\xi_3 = \sqrt{n-1}(1-p_0)/n$, $\eta_1 = (\sqrt{3}/2)(1 - (n+1)(1-p_0)/n)$ and $\eta_2 = (1/2)(1 - 3(n-1)(1-p_0)/n)$. Thus these five points form a hyper-polyhedron in 8-dimensional space. Then all rank-3 quantum states corresponding to the points in this hyper-polyhedron have vanishing three-tangle.

Now we would like to consider the Coffman-Kundu-Wootters(CKW) relation[71], which is

$$4\det\rho_A = \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 + \tau_3(\psi) \tag{7.24}$$

[1] Unlike qubit system not all points in S^8 do correspond to the qutrit states due to the condition of star product[138]

for three-qubit pure state $|\psi\rangle$. In Eq.(7.24) $\mathcal{C}_{AB}$ and $\mathcal{C}_{AC}$ are the concurrences for the corresponding reduced states. Eq.(7.24) indicates that the entanglement of qubit A is originated from both bipartite and tripartite contributions. For mixed state Ref.[71] has shown

$$4 \min \left[\det(\rho_A)\right] \geq \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2, \tag{7.25}$$

where minimum of one-tangle is taken over all possible decompositions of ρ. In Ref.[76] the CKW inequality (7.25) has been examined for the mixture of GHZ and W states. For this case it was shown that the one-tangle is always larger than the sum of squared concurrences and three-tangle.

Now, we would like to check the CKW inequality for $\rho(p,q)$ in Eq.(7.8) with $q = (1-p)/n$. In this case one can compute the minimum one-tangle directly, whose expression is

$$\begin{aligned} 4 \min \left[\det\rho_A\right] = \frac{1}{9}\Big[& (8 - 4p - 12q + 5p^2 + 12q^2 + 12pq) \qquad (7.26) \\ & +4\sqrt{pq(1-p-q)}\left(2\sqrt{6q} + 2\sqrt{6(1-p-q)} - 3\sqrt{p}\right)\Big]. \end{aligned}$$

Also it is straightforward to compute the sum of squared concurrences, which is

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 = 2\left(\max\left[0, \frac{2}{3}(1-p) - \frac{1}{3}\sqrt{(3p+2q)(2+p-2q)}\right]\right)^2. \tag{7.27}$$

The one-tangle(upper solid lines), $\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2$(left solid lines), and three-tangle(right solid lines) are plotted in Fig. 7.1 for $n = 1$, $n = 2$ and $n = 10$. This figure indicates that all quantities approach to their corresponding $n = 1$ quantity when n increases from $n = 2$. This is consistent with the fact that $\rho(p,q)$ with $n = 1$ is LU-equivalent to $\rho(p,q)$ with $n = \infty$. The inequality

$$4 \min \left[\det(\rho_A)\right] \geq \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 + \tau_3 \tag{7.28}$$

holds for all n. In the region $p_C \leq p \leq p_0$, where

$$p_C = \frac{(7n^2 - 4n + 4) - 3n\sqrt{5n^2 - 4n + 4}}{(n-2)^2}, \tag{7.29}$$

both $\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2$ and τ_3 vanish while there is quite substantial one-tangle. Its interpretation is given in Ref.[76] from the mathematical point of view. However, its physical meaning is still unclear at least for us. In the region $p \geq p_C$ and $p \leq p_0$ the entanglement of the qubit A mainly stems from the bipartite and tripartite correlations, respectively.

One may wonder why we do not take $q = \alpha(1-p)$ with real number $0 \leq \alpha \leq 1$. For this case, however, it is unclear whether or not the p-dependence of $\tau_3(p, q, \varphi_1, \varphi_2)$ in Eq.(7.12) has maximum zero at $\varphi_1 = \varphi_2 = 0$ regardless of α. If this is correct, our result can be easily extended to the case of $q = \alpha(1-p)$ by changing $n \rightarrow 1/\alpha$.

There are many rank-3 mixed states whose three-tangles may exhibit interesting behavior. For example, let us consider the state

$$\pi(p,n) = p|GHZ,+\rangle\langle GHZ,+| + \frac{1-p}{n}|W\rangle\langle W| + \frac{(n-1)(1-p)}{n}|GHZ,-\rangle\langle GHZ,-|, \tag{7.30}$$

where $|GHZ,\pm\rangle = (1/\sqrt{2})(|000\rangle \pm |111\rangle)$. Unlike $\rho(p,n)$ discussed in the present chapter $\pi(p,1)$ is not LU-equivalent with $\pi(p,\infty)$. When $n=1$, $\pi(p,1)$ is identical with $\rho(p,q)$ with $n=1$. When $n=\infty$, the three-tangle of $\pi(p,\infty)$ can be calculated by similar method and the result is $(2p-1)^2$. If n increases from $n=2$, the three-tangle should move to $(2p-1)^2$ from Eq.(7.6) smoothly. The particular point $p=1/2$ may play a role as a fixed point. It is interesting to examine this behavior by deriving the optimal decomposition of $\pi(p,n)$ in the full range of p and n.

Of course, it is extremely important if we develop a calculational technique, which enables us to compute the three-tangle for the arbitrary mixed states. In order to explore this issue we should develop a technique first, which enables us to compute the three-tangle for the arbitrary rank-two mixed states as Hill and Wootters did in the concurrence calculation in Ref.[38]. For the case of concurrence, however, Hill and Wootters exploited fully the magic properties of the magic basis $\{|e_i\rangle, i = 1, \cdots, 4\}$. In this basis the concurrence for the two-qubit state $|\psi\rangle$ can be expressed as $|\sum \alpha_i^2|$, where $|\psi\rangle = \sum_i \alpha_i |e_i\rangle$. Then this property and usual convexification technique make it possible to compute the concurrence for the arbitrary rank-two bipartite mixed states. Such a basis, however, is not found in the three-qubit system so far. Furthermore, we do not know whether or not such a basis exists in the higher-qubit system. Thus it is very difficult problem to go further this issue.

From the aspect of physics it is also of interest to investigate the physical role of the three-tangle. As shown in Ref.[57] the two-qubit mixed-state entanglement provides an information on the fidelity in the bipartite teleportation through noisy channels. Since the three-tangle is purely tripartite entanglement, it may give certain information in the scheme of quantum copy machine or three-party quantum teleportation[139]. It seems to be interesting to explore the physical role of the three-tangle in the particular real tasks.

Chapter 8

Three-tangle does not properly quantify tripartite entanglement for Greenberger-Horne-Zeilinger-type states

Nowadays, it is well-known that entanglement is the most valuable physical resource for the quantum information processing such as quantum teleportation[19], superdense coding[20], quantum cloning[140], quantum algorithms[22, 24], quantum cryptography[125], and quantum computer technology[32, 15]. Thus, it is highly important to understand the various properties of the mutipartite entanglement of the quantum states.

The main obstacle for characterizing the entanglement of the multipartite state is its calculational difficulties even if original definition of the entanglement measure itself is comparatively simple. In addition, computation of the entanglement for the multipartite mixed states is much more difficult than that for the pure states, mainly due to the fact that the entanglement for the mixed states, in general, is defined by a convex-roof extension[36, 137]. In order to compute the entanglement explicitly for the mixed states, therefore, we should find an optimal decomposition of the given mixed state, which provides a minimum value of the entanglement over all possible ensembles of pure states. However, there is no general way for finding the optimal decomposition for the arbitrary mixed states except bipartite cases[38, 39]. Thus, it becomes a central issue for the computation of the mixed state entanglement.

Few years ago, fortunately, Wootters[38, 39] has shown how to construct the optimal decompositions for the most simple bipartite cases. This enables us to be able to compute the concurrence, one of the entanglement measure, analytically for the arbitrary 2-qubit mixed states. It also makes it possible to understand more deeply the role of the entanglement in the real quantum information processing[57]. Most importantly, it becomes a basis for the quantification of three-party entanglement called residual entanglement or three-tangle[71]. Thus, it is extremely important to find a calculation tool for the three-tangle if one wants to take a step toward a fundamental issue, i.e. characterization of the mutipartite mixed state entanglement.

It is well-known[65] that the three-qubit pure states can be classified by product states $(A-B-C)$, biseparable states $(A-BC, B-AC, C-AB)$ and true tripartite states (ABC) through stochastic local operation and classical communication(SLOCC). The true tripartite states consist of two different classes, GHZ-class and W-class, where

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \\ |W\rangle &= \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right). \end{aligned} \tag{8.1}$$

Since the three-tangle τ_3 for the pure state $|\psi\rangle = \sum_{i,j,k=0}^{1} a_{ijk}|ijk\rangle$ is defined as[71]

$$\tau_3 = 4|d_1 - 2d_2 + 4d_3| \tag{8.2}$$

with

$$\begin{aligned} d_1 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\ d_2 &= a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} \\ &\quad + a_{000}a_{111}a_{110}a_{001} + a_{011}a_{100}a_{101}a_{010} \\ &\quad + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001} \\ d_3 &= a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}, \end{aligned} \tag{8.3}$$

it is easy to show that the product and biseparable states have zero three-tangle. This fact implies that the three-tangle is a genuine measure for the three-party entanglement.

However, there is a crucial defect in the three-tangle as a three-party entanglement measure. While the three-tangle for the GHZ state is maximal, i.e. $\tau_3(GHZ) = 1$, it vanishes for the W-state. This means that the three-tangle does not properly quantify the three-party entanglement for the W-type states. The purpose of this chapter is to show that besides W-type states the three-tangle τ_3 does not properly quantify the three-party entanglement for a rank-3 mixtures composed of only three GHZ-type states.

Recently, the three-tangle for rank-2 mixture of GHZ and W states is analytically computed[76, 78]. In Ref.[80], furthermore, the three-tangle for the rank-3 mixture of GHZ, W, and inverted W states is also analytically computed. In this chapter we start with showing that a mixed state

$$\Pi_{GHZ} = \frac{1}{3}\left[\begin{array}{l} |GHZ,2+\rangle\langle GHZ,2+| + |GHZ,3+\rangle\langle GHZ,3+| \\ + |GHZ,4+\rangle\langle GHZ,4+| \end{array}\right] \tag{8.4}$$

has vanishing three-tangle, where we define for later use as following:

$$\begin{array}{l} |GHZ,1\pm\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle \pm |111\rangle\right) \\ |GHZ,2\pm\rangle = \frac{1}{\sqrt{2}}\left(|001\rangle \pm |110\rangle\right) \\ |GHZ,3\pm\rangle = \frac{1}{\sqrt{2}}\left(|010\rangle \pm |101\rangle\right) \\ |GHZ,4\pm\rangle = \frac{1}{\sqrt{2}}\left(|011\rangle \pm |100\rangle\right). \end{array} \tag{8.5}$$

Let us consider a pure state

$$\begin{array}{rl} |J(\theta_1,\theta_2)\rangle = & \frac{1}{\sqrt{3}}|GHZ,2+\rangle - \frac{1}{\sqrt{3}}e^{i\theta_1}|GHZ,3+\rangle \\ & -\frac{1}{\sqrt{3}}e^{i\theta_2}|GHZ,4+\rangle. \end{array} \tag{8.6}$$

Then, it is easy to show that the three-tangle of $|J(\theta_1,\theta_2)\rangle$ is

$$\tau_3(\theta_1,\theta_2) = \frac{1}{9}|1-\left(e^{i\theta_1}-e^{i\theta_2}\right)^2||1-\left(e^{i\theta_1}+e^{i\theta_2}\right)^2|, \tag{8.7}$$

which vanishes when

$$(\theta_1,\theta_2) = \left\{\begin{array}{l} (\ \pi/3, 2\pi/3), (5\pi/3, 4\pi/3) \\ (2\pi/3,\ \pi/3), (4\pi/3, 5\pi/3) \\ (\ \pi/3, 5\pi/3), (5\pi/3,\ \pi/3) \\ (2\pi/3, 4\pi/3), (4\pi/3, 2\pi/3) \end{array}\right\}. \tag{8.8}$$

Moreover, one can show straightforwardly that Π_{GHZ} can be decomposed into

$$\Pi_{GHZ} = \frac{1}{8}\left[\begin{array}{l} |J(\pi/3,2\pi/3)\rangle\langle J(\pi/3,2\pi/3)| + |J(5\pi/3,4\pi/3)\rangle\langle J(5\pi/3,4\pi/3)| \\ + |J(\pi/3,5\pi/3)\rangle\langle J(\pi/3,5\pi/3)| + |J(2\pi/3,4\pi/3)\rangle\langle J(2\pi/3,4\pi/3)| \\ + \text{terms with exchanged arguments} \end{array}\right]. \tag{8.9}$$

Combining Eq.(8.8) and (8.9), one can show that Eq.(8.9) is the optimal decomposition of Π_{GHZ} and its three-tangle is zero:

$$\tau_3 \left(\Pi_{GHZ}\right) = 0. \tag{8.10}$$

The reason why Π_{GHZ} has vanishing three-tangle is that the optimal ensembles given in Eq. (8.9) are all W-states. Therefore, Π_{GHZ} can also be expressed in terms of only W-states. As a result, we encounter a very strange situation that Π_{GHZ} has vanishing three- and two-tangles[1], but non-vanishing one-tangle

$$4 \min \left[\det \left(\mathrm{Tr}_{BC} \Pi_{GHZ}\right)\right] = \frac{5}{9}. \tag{8.11}$$

For comparison one can compute π-tangle[141], another three-party entanglement measure defined in terms of the global negativities[142]. It is easy to show that the π-tangle of Π_{GHZ} is not vanishing but 1/9. This fact seems to indicate that the three-tangle does not properly reflect the three-party entanglement for GHZ-type states as well as W-type states.

We can use Eq.(8.10) for computing the three-tangles of the higher-rank mixed states. For example, let us consider the following rank-4 state

$$\sigma = x|GHZ, 1+\rangle\langle GHZ, 1+| + (1-x)\Pi_{GHZ} \tag{8.12}$$

with $0 \leq x \leq 1$. In order to compute the three-tangles for σ we first consider a pure state

$$|X(x, \varphi_1, \varphi_2, \varphi_3)\rangle = \sqrt{x}|GHZ, 1+\rangle - \sqrt{\frac{1-x}{3}} \begin{pmatrix} e^{i\varphi_1}|GHZ, 2+\rangle + e^{i\varphi_2}|GHZ, 3+\rangle \\ +e^{i\varphi_3}|GHZ, 4+\rangle \end{pmatrix}. \tag{8.13}$$

Then it is easy to show that the three-tangle of $|X(x, \varphi_1, \varphi_2, \varphi_3)\rangle$ becomes

$$\tau_3 \left(|X(x, \varphi_1, \varphi_2, \varphi_3)\rangle\right) = \left| \begin{array}{l} x^2 + \frac{(1-x)^2}{9} \left(e^{4i\varphi_1} + e^{4i\varphi_2} + e^{4i\varphi_3}\right) \\ -\frac{2}{3}x(1-x) \left(e^{2i\varphi_1} + e^{2i\varphi_2} + e^{2i\varphi_3}\right) \\ -\frac{2}{9}(1-x)^2 \begin{pmatrix} e^{2i(\varphi_1+\varphi_2)} + e^{2i(\varphi_1+\varphi_3)} \\ +e^{2i(\varphi_2+\varphi_3)} \end{pmatrix} \\ -\frac{8\sqrt{3}}{9} \sqrt{x(1-x)^3} e^{i(\varphi_1+\varphi_2+\varphi_3)} \end{array} \right|. \tag{8.14}$$

The vectors $|X(x, \varphi_1, \varphi_2, \varphi_3)\rangle$ has following properties. The three-tangle of it has the largest zero at $x = x_0 \equiv 3/4$ and $\varphi_1 = \varphi_2 = \varphi_3 = 0$. The vectors $|X(x, 0, 0, 0)\rangle$,

[1]It is easy to show that $\mathcal{C}_{AB}^2$ and $\mathcal{C}_{AC}^2$ are zero, where $\mathcal{C}$ is concurrence for corresponding reduced states.

$|X(x,0,\pi,\pi)\rangle$, $|X(x,\pi,0,\pi)\rangle$ and $|X(x,\pi,\pi,0)\rangle$ have same three-tangles. Finally, σ can be decomposed into

$$\sigma = \frac{1}{4}\left[\begin{array}{l} |X(x,0,0,0)\rangle\langle X(x,0,0,0)| + |X(x,0,\pi,\pi)\rangle\langle X(x,0,\pi,\pi)| \\ +|X(x,\pi,0,\pi)\rangle\langle X(x,\pi,0,\pi)| + |X(x,\pi,\pi,0)\rangle\langle X(x,\pi,\pi,0)| \end{array}\right]. \tag{8.15}$$

When $x \leq x_0$, one can construct the optimal decomposition in the following form:

$$\begin{aligned}\sigma &= \frac{x}{4x_0}\left[\begin{array}{l} |X(x_0,0,0,0)\rangle\langle X(x_0,0,0,0)| + |X(x_0,0,\pi,\pi)\rangle\langle X(x_0,0,\pi,\pi)| \\ +|X(x_0,\pi,0,\pi)\rangle\langle X(x_0,\pi,0,\pi)| + |X(x_0,\pi,\pi,0)\rangle\langle X(x_0,\pi,\pi,0)| \end{array}\right] \\ &\quad + \frac{x_0 - x}{x_0}\Pi_{GHZ}. \end{aligned} \tag{8.16}$$

Since Π_{GHZ} has the vanishing three-tangle, one can show easily

$$\tau_3(\sigma) = 0 \qquad \text{when } \ x \leq x_0 = 3/4. \tag{8.17}$$

Now, let us consider the three-tangle of σ in the region $x_0 \leq x \leq 1$. Since Eq.(8.15) is an optimal decomposition at $x = x_0$, one can conjecture that it is also optimal in the region $x_0 \leq x$. As will be shown shortly, however, this is not true at the large-x region. If we compute the three-tangle under the condition that Eq.(8.15) is optimal at $x_0 \leq x$, its expression becomes

$$\alpha_I(x) = x^2 - \frac{1}{3}(1-x)^2 - 2x(1-x) - \frac{8\sqrt{3}}{9}\sqrt{x(1-x)^3}. \tag{8.18}$$

However, one can show straightforwardly that $\alpha_I(x)$ is not a convex function in the region $x \geq x_*$, where

$$x_* = \frac{1}{4}\left(1 + 2^{1/3} + 4^{1/3}\right) \approx 0.961831. \tag{8.19}$$

Therefore, we need to convexify $\alpha_I(x)$ in the region $x_1 \leq x \leq 1$ to make the three-tangle to be convex function, where x_1 is some number between x_0 and x_*. The number x_1 will be determined shortly.

In the large x-region one can derive the optimal decomposition in a form:

$$\begin{aligned}\sigma &= \frac{1-x}{4(1-x_1)}\left[\begin{array}{l} |X(x_1,0,0,0)\rangle\langle X(x_1,0,0,0)| + |X(x_1,0,\pi,\pi)\rangle\langle X(x_1,0,\pi,\pi)| \\ +|X(x_1,\pi,0,\pi)\rangle\langle X(x_1,\pi,0,\pi)| + |X(x_1,\pi,\pi,0)\rangle\langle X(x_1,\pi,\pi,0)| \end{array}\right] \\ &\quad + \frac{x - x_1}{1 - x_1}|GHZ, 1+\rangle\langle GHZ, 1+| \end{aligned} \tag{8.20}$$

which gives a three-tangle as

$$\alpha_{II}(x, x_1) = \frac{1-x}{1-x_1}\alpha_I(x_1) + \frac{x - x_1}{1 - x_1}. \tag{8.21}$$

Since $d^2\alpha_{II}/dx^2 = 0$, there is no convex problem if $\alpha_{II}(x, x_1)$ is a three-tangle in the large-x region. The constant x_1 can be fixed from the condition of minimum α_{II}, i.e. $\partial\alpha_{II}(x, x_1)/\partial x_1 = 0$, which gives

$$x_1 = \frac{1}{4}(2 + \sqrt{3}) \approx 0.933013. \tag{8.22}$$

As expected, x_1 is between x_0 and x_*. Thus, finally the three-tangle for σ becomes

$$\tau_3(\sigma) = \begin{cases} 0 & x \leq x_0 \\ \alpha_I(x) & x_0 \leq x \leq x_1 \\ \alpha_{II}(x, x_1) & x_1 \leq x \leq 1 \end{cases} \tag{8.23}$$

and the corresponding optimal decompositions are Eq.(8.16), Eq.(8.15) and Eq.(8.20) respectively. In order to show Eq.(8.23) is genuine optimal, first we plot x-dependence of Eq.(8.14) for various φ_i $(i = 1, 2, 3)$. These curves have been referred as the characteristic curves[79]. Then, one can show, at least numerically, that Eq.(8.23) is a convex hull of the minimum of the characteristic curves, which implies that Eq.(8.23) is genuine three-tangle for σ.

It is straightforward to show that the mixture σ has vanishing two-tangles, i.e. $\mathcal{C}_{AB} = \mathcal{C}_{AC} = 0$, but non-vanishing one-tangle

$$\mathcal{C}^2_{A(BC)}(\sigma) = \frac{1}{9}\left(5 - 4x + 8x^2 - 8\sqrt{3x(1-x)^3}\right). \tag{8.24}$$

Thus, the monogamy inequality $\tau_3 + \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} \leq \mathcal{C}^2_{A(BC)}$ holds for the rank-4 mixture σ.

Eq.(8.10) can be used to compute the upper bound of the three-tangle for the higher-rank states. For example, let us consider the following rank-8 state

$$\rho = \xi\sigma + (1 - \xi)\tilde{\sigma} \tag{8.25}$$

where σ is given in Eq.(8.12) and $\tilde{\sigma}$ is

$$\begin{aligned} \tilde{\sigma} = y|GHZ, 1-\rangle\langle GHZ, 1-| \\ + \tfrac{1-y}{3}\left[\begin{array}{c} |GHZ, 2-\rangle\langle GHZ, 2-| + |GHZ, 3-\rangle\langle GHZ, 3-| \\ + |GHZ, 4-\rangle\langle GHZ, 4-| \end{array}\right]. \end{aligned} \tag{8.26}$$

If $x = y$, σ and $\tilde{\sigma}$ are local-unitary(LU) equivalent with each other. Since the three-tangle is LU-invariant quantity, $\tau_3(\tilde{\sigma})$ should be identical to $\tau_3(\sigma)$ when $x = y$

Since ρ is rank-8 mixed state, it seems to be extremely difficult to compute its three-tangle analytically. If, however, $0 \leq y \leq 3/4$, $\tau_3(\tilde{\sigma})$ becomes zero and the above analysis yields a non-trivial upper bound of $\tau_3(\rho)$ as following:

$$\tau_3(\rho) \leq \xi\tau_3(\sigma). \tag{8.27}$$

In this chapter we have shown that the three-tangle does not properly quantify the three-party entanglement for some mixture composed of only GHZ states. This fact has been used to compute the (upper bound of) three-tangles for the higher-rank mixed states.

The fact $\tau_3(\sigma) = 0$ for $x \leq 3/4$ can be used to find other rank-4 mixtures which have vanishing three-tangle by considering the Bloch hypersphere of $d = 4$ qudit system. First, we correspond the GHZ-states in σ to the basis of the qudit system as follows:

$$\begin{array}{l} |GHZ,1+\rangle = (1,0,0,0)^T \, , \; |GHZ,2+\rangle = (0,1,0,0)^T \\ |GHZ,3+\rangle = (0,0,1,0)^T \, , \; |GHZ,4+\rangle = (0,0,0,1)^T \end{array} \tag{8.28}$$

where T stands for transposition. It is well-known[143] that the density matrix of the arbitrary $d = 4$ qudit state can be represented by $\rho = (1/4)(I + \sqrt{6}\vec{n} \cdot \vec{\lambda})$, where $\vec{n}$ is 15-dimensional unit vector and

$$\vec{\lambda} = \left(\Lambda_s^{12}, \cdots, \Lambda_s^{34}, \Lambda_a^{12}, \cdots, \Lambda_a^{34}, \Lambda^1, \Lambda^2, \Lambda^3\right). \tag{8.29}$$

The generalized Gell-Mann matrices Λ_s^{ij}, Λ_a^{ij} and Λ^j are explicitly given in Ref.[143]. Then, the 15-dimensional Bloch vectors for $|X\,(3/4,0,0,0)\rangle$, $|X\,(3/4,0,\pi,\pi)\rangle$, $|X\,(3/4,\pi,0,\pi)\rangle$, and $|X\,(3/4,\pi,\pi,0)\rangle$ can be easily derived. Thus, these four points form a hyper-polyhedron in 16-dimensional space. Then all rank-4 quantum states corresponding to the points in this hyper-polyhedron have vanishing three-tangle.

As we have shown in this chapter, Π_{GHZ} has vanishing two and three-tangle, but non-vanishing one-tangle. It makes the left-hand side of the monogamy inequality $\tau_3 + \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \leq \mathcal{C}_{A(BC)}^2$ reduce zero. Thus, natural question arises: what physical resources make the one-tangle to be non-vanishing? Authors in Ref.[144] conjectured that the origin of the non-vanishing one-tangle comes from the higher tangles of the purified state. To support their argument they considered a multipartite entanglement measure defined

$$E_{ms}(\Psi_N) = \frac{\sum_k \tau_{k(R_k)} - 2\sum_{i<j} \mathcal{C}_{ij}^2}{N} \tag{8.30}$$

where $\tau_{k(R_k)} = 2(1 - \mathrm{Tr}\rho_k^2)$ and $|\Psi_N\rangle$ is a N-qubit purified state of the given mixed state. Since the numerator of E_{ms} is difference between the total one-tangle and total two-tangle, it measures a contribution of the higher-tangles to the one-tangle. If we choose the purified state as

$$|\Psi_5\rangle = \frac{1}{\sqrt{3}}\left(\,|GHZ,2+\rangle|00\rangle + |GHZ,3+\rangle|01\rangle + |GHZ,4+\rangle|10\rangle\,\right), \tag{8.31}$$

$E_{ms}(\Psi_5)$ reduces to 43/45, which is larger than the one-tangle 5/9. Thus, it is possible that part of $E_{ms}(\Psi_5)$ converts into the non-vanishing one-tangle. However, still we do not know how to compute the one-tangle explicitly from $E_{ms}(\Psi_5)$.

The three-tangle itself is a good three-party entanglement measure. It exactly coincides with the modulus of a Cayley's hyperdeterminant[72, 73] and is polynomial invariant under the local $SL(2, \mathbb{C})$ transformation[74, 75]. As shown, however, it cannot properly quantify the three-party entanglement of W-state and Π_{GHZ}: $\tau_3(W) = \tau_3(\Pi_{GHZ}) = 0$. On the other hand, the π-tangle gives the non-zero values: $\pi_3(W) = 4(\sqrt{5} - 1)/9$ and $\pi_3(\Pi_{GHZ}) = 1/9$. Does this fact simply imply the crucial defects of the three-tangle as a three-party entanglement measure? Here, we would like to comment on the physical implication of $\tau_3(\Pi_{GHZ}) = 0$. Few years ago the three-qubit mixed states were classified in Ref.[81]. Following Ref.[81] the whole mixed states are classified as separable (S), biseparable (B), W and GHZ classes. These classes satisfy $S \subset B \subset W \subset GHZ$. One remarkable fact, which was proved in this reference, is that the $W \setminus B$ class is not of measure zero among all mixed-states. This is contrary to the case of the pure states, where the set of W-state forms measure zero[65]. This fact implies that the portion of $W \setminus B$ class in the whole mixed states becomes larger compared to that of W class in the whole pure states. How could this happen? The fact $\tau_3(\Pi_{GHZ}) = 0$ sheds light on this issue. Since Π_{GHZ} has zero three-tangle but non-zero π-tangle, it is manifestly an element of $W \setminus B$ class. As shown in Eq.(8.4), however, it consists of three GHZ states without pure W-type state. We think there are many $W \setminus B$ states, which are mixture of only GHZ states. It increases the portion of $W \setminus B$ class and eventually makes the $W \setminus B$ class to be of non-zero measure in the whole mixed states.

Chapter 9

Conclusion

In this thesis we present the recent results on the computation of the pure- and mixed-states entanglement measures. As well-known, entanglement is a genuine physical resource for the quantum information processing. In fact, entanglement plays a central role in teleportation process[19, 57, 139, 145, 115], superdense coding[20, 145, 115], quantum copy machine[146, 147, 148, 149], quantum cryptography[18, 21, 150, 151] *etc*. In this reason it is highly important to understand and characterize the entanglement of the quantum states.

In chapter II we have presented the three theorems which is important for the computation of the geometric and Groverian entanglement measure. Theorem 1 enables us to compute the entanglement measures if we know only the one of the 1-particle reduced states obtained by taking partial trace. This theorem is more useful for the computation of the entanglement of the 3-qubit states because it makes it possible to compute the measures with the two qubit reduced state. Theorem 2 says that if the 1-particle reduced states of the two pure-states are LU-equivalent, these states have same geometric and Groverian entanglement measures. Therefore, this theorem can be used to characterize the set of the pure states. Theorem 3 states the upper bound of the entanglement measure. This theorem can be used to characterize the maximally entangled states in the higher-qubit system.

Using a Theorem 1 given in chapter 2 we have computed the geometric measure defined $\Lambda(\psi) = 1 - P_{max}(\psi)$ or Groverian entanglement measure defined $G(\psi) = \sqrt{1 - P_{max}(\psi)}$ for the generalized W-state

$$|W\rangle = a|100\rangle + b|010\rangle + c|001\rangle, \tag{9.1}$$

where P_{max} is a maximal overlap probability with a set of the separable states. It turns out that P_{max} for W-state has two different expressions depending on the parameter space. To summarize the result of this chapter we define $\{\alpha, \beta, \gamma\}$ as $\{a, b, c\}$ with decreasing order. If, then, $\alpha^2 \geq \beta^2 + \gamma^2$, $P_{max} = \max(a^2, b^2, c^2) = \alpha^2$ and if $\alpha^2 \leq \beta^2 + \gamma^2$, $P_{max} = 4R^2$, where

$$R = \frac{abc}{\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}} \tag{9.2}$$

is a circumradius of the triangle (a, b, c). This is a first report on the fact that the entanglement measure can be expressed in terms of the geometrical quantities of the polygons.

To be sure that the expression of the entanglement measure in terms of the geometrical quantities is a general property we have considered the more general 3-qubit state

$$|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle + d|111\rangle \tag{9.3}$$

in chapter 3. It turns out that for this state P_{max} has generally three different expressions. However, one of them, expression in terms of the circumradius of the crossed quadrangle does not have applicable domain in the parameter space. As a result, P_{max} for $|\psi\rangle$ has two different expression depending on the parameter space. To summarize it we define $\{\alpha, \beta, \gamma, \delta\}$ as $\{a, b, c, d\}$ with decreasing order. If $\alpha^2 \geq \beta^2 + \gamma^2 + \delta^2 + 2\beta\gamma\delta/\alpha$, then $P_{max} = \max(a^2, b^2, c^2, d^2) = \alpha^2$ and $\alpha^2 \leq \beta^2 + \gamma^2 + \delta^2 + 2\beta\gamma\delta/\alpha$, $P_{max} = 4R^2$, where

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(p-a)(p-b)(p-c)(p-d)}} \tag{9.4}$$

is a circumradius of a convex quadrangle (a, b, c, d), where $p = (a+b+c+d)/2$. Therefore, the expression of the entanglement in terms of some geometrical quantities seems to be general property of the entanglement.

Since the entanglement measures should be entanglement monotone and LU-invariant, it is of great interest to express the derived results in terms of the LU-invariants. Few years ago Acin *et al*[68] classified the whole set of 3-qubit pure states as a five types via the generalized Schmidt decomposition. Using these classifications we have computed the entanglement measure for type I, type II and type III and re-expressed them in terms of the LU-invariants J_i $(i = 1, \cdots, 5)$ in chapter V. Although we have failed to derive the analytic expressions of the entanglement measure for type IV and type V, we have shown that the phase factor of the quantum state does not have any effect to the entanglement measure for type IV. This fact with geometric interpretation on the entanglement enables us to derive the analytical expression for the general arbitrary three-qubit states in the future.

The successful geometrical interpretation for the geometric or Groverian entanglement measure for the three-qubit states as presented in chapter III and chapter IV gives rise to a following question naturally: does the entanglement of the higher-qubit system also have some geometrical meaning? This question is explored in chapter VI by considering the one-parametric n-qubit W-state and two-parametric 4-qubit W-state. Although we have derived the analytic expressions of the entangled measures for these quantum states, we have failed to find a geometrical interpretation because we do not have complete expression for the generalized W-states. Still, it is open problem to find a connection between geometry of polygons and multipartite entanglement. We hope we understand more profoundly the meaning and physical implication of this connection.

In chapter VII and VIII we have considered the entanglement for the mixed states. Since the entanglement for the mixed state is generally defined by a convex-roof extension, it is much more difficult to compute the entanglement measure for the mixed state because it is highly non-trivial to find a optimal decomposition for the given mixed state. In chapter VI we have computed the three-tangle for the mixture composed of GHZ, W and flipped W states, whose density matrix is

$$\rho(p,q) = p|GHZ\rangle\langle GHZ| + q|W\rangle\langle W| + (1-p-q)|\tilde{W}\rangle\langle\tilde{W}| \tag{9.5}$$

where $|GHZ\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$, $|W\rangle = (1/\sqrt{3})(|100\rangle + |010\rangle + |001\rangle)$ and $|\tilde{W}\rangle = (1/\sqrt{3})(|011\rangle + |101\rangle + |110\rangle)$. It turns out that depending on p and q the three-tangle of $\rho(p,q)$ has three different expressions. We also provide an analytical technique, which determines whether or not an arbitrary rank-3 mixed state has vanishing three-tangle by making use of the Bloch sphere S^8 of the qutrit system. We also briefly discussed in this chapter the physical implication of the monogamy inequality, generalization of CKW-inequality derived in Ref.[71].

It is well-known that W-state has vanishing three-tangle even though W-state has a pure tripartite entanglement. This means that the three-tangle does not properly quantify the tripartite entanglement of W-state. In chapter VIII in addition to W-state we have shown that the three-tangle does not properly quantify the tripartite entanglement of some mixed states composed of only GHZ states. To verify this statement explicitly we have constructed a mixture defined

$$\Pi_{GHZ} = \frac{1}{3}\Big[|GHZ,2+\rangle\langle GHZ,2+| + |GHZ,3+\rangle\langle GHZ,3+| + |GHZ,4+\rangle\langle GHZ,4+|\Big] \tag{9.6}$$

where

$$|GHZ, 2\pm\rangle = \frac{1}{\sqrt{2}}\left(|001\rangle \pm |110\rangle\right) \qquad |GHZ, 3\pm\rangle = \frac{1}{\sqrt{2}}\left(|010\rangle \pm |101\rangle\right)$$
$$|GHZ, 4\pm\rangle = \frac{1}{\sqrt{2}}\left(|011\rangle \pm |100\rangle\right), \tag{9.7}$$

and have shown that Π_{GHZ} has a vanishing three-tangle. This surprising fact indicates that there should exist an pure state ensemble of Π_{GHZ}, whose pure-states are all W-states even though it is composed of only GHZ states. Such an ensemble is explicitly constructed in this chapter. This fact may shed light on the physical reason why the set of the mixed W-states is not of measure zero in the whole set of the three-qubit mixed states while pure W-states have zero measure.

Quantum information theories have various applicable regions such as quantum computer, quantum cryptography, quantum copying, quantum repeater, *etc.* Above all my major concern is to apply the quantum information techniques to the black hole physics, especially the information loss via Hawking radiation. In the future I would like to explore this subject and contribute to the development of quantum gravity.

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Appendix A

One can easily show that the elements of $\mathcal{O}$ defined in Eq.(5.6) are given by

$$
\begin{aligned}
\mathcal{O}_{11} &= \frac{1}{2}\left(u_{11}u_{22}^* + u_{11}^*u_{22} + u_{12}u_{21}^* + u_{12}^*u_{21}\right) \\
\mathcal{O}_{22} &= \frac{1}{2}\left(u_{11}u_{22}^* + u_{11}^*u_{22} - u_{12}u_{21}^* - u_{12}^*u_{21}\right) \\
\mathcal{O}_{33} &= |u_{11}|^2 - |u_{12}|^2 \\
\mathcal{O}_{12} &= \frac{i}{2}\left(u_{12}u_{21}^* + u_{11}u_{22}^* - u_{12}^*u_{21} - u_{11}^*u_{22}\right) \\
\mathcal{O}_{21} &= \frac{i}{2}\left(u_{12}u_{21}^* + u_{11}^*u_{22} - u_{12}^*u_{21} - u_{11}u_{22}^*\right) \\
\mathcal{O}_{13} &= u_{11}u_{12}^* + u_{11}^*u_{12} \\
\mathcal{O}_{31} &= u_{11}u_{21}^* + u_{11}^*u_{21} \\
\mathcal{O}_{23} &= -i\left(u_{11}u_{12}^* + u_{21}^*u_{22}\right) \\
\mathcal{O}_{32} &= i\left(u_{11}u_{21}^* + u_{12}^*u_{22}\right)
\end{aligned}
\tag{A.1}
$$

where u_{ij} is element of the unitary matrix defined in Eq.(5.6). It is easy to prove $\mathcal{O}\mathcal{O}^T = \mathcal{O}^T\mathcal{O} = \mathbb{1}$, which indicates that $\mathcal{O}_{\alpha\beta}$ is an element of O(3).

Appendix B

If the density matrix associated from the pure state $|\psi\rangle$ in Eq.(5.12) is represented by Bloch form like Eq.(5.11), the explicit expressions for $\vec{v}_i$ are

$$\vec{v}_1 = \begin{pmatrix} 2\lambda_0\lambda_1\cos\varphi \\ 2\lambda_0\lambda_1\sin\varphi \\ \lambda_0^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 2\lambda_1\lambda_3\cos\varphi + 2\lambda_2\lambda_4 \\ -2\lambda_1\lambda_3\sin\varphi \\ \lambda_0^2 + \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \end{pmatrix}$$
$$\vec{v}_3 = \begin{pmatrix} 2\lambda_1\lambda_2\cos\varphi + 2\lambda_3\lambda_4 \\ -2\lambda_1\lambda_2\sin\varphi \\ \lambda_0^2 + \lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2 \end{pmatrix} \tag{B.1}$$

and the components of $h^{(i)}$ are

$$\begin{aligned}
&h_{11}^{(1)} = 2\lambda_2\lambda_3 + 2\lambda_1\lambda_4\cos\varphi, \qquad h_{22}^{(1)} = 2\lambda_2\lambda_3 - 2\lambda_1\lambda_4\cos\varphi \\
&h_{33}^{(1)} = \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2, \qquad h_{12}^{(1)} = h_{21}^{(1)} = -2\lambda_1\lambda_4\sin\varphi \\
&h_{13}^{(1)} = -2\lambda_2\lambda_4 + 2\lambda_1\lambda_3\cos\varphi, \qquad h_{31}^{(1)} = -2\lambda_3\lambda_4 + 2\lambda_1\lambda_2\cos\varphi \\
&h_{23}^{(1)} = -2\lambda_1\lambda_3\sin\varphi, \qquad h_{32}^{(1)} = -2\lambda_1\lambda_2\sin\varphi \\
&h_{11}^{(2)} = -h_{22}^{(2)} = 2\lambda_0\lambda_2, \qquad h_{33}^{(2)} = \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 + \lambda_4^2 \\
&h_{12}^{(2)} = h_{21}^{(2)} = 0, \qquad h_{13}^{(2)} = 2\lambda_0\lambda_1\cos\varphi \\
&h_{31}^{(2)} = -2\lambda_3\lambda_4 - 2\lambda_1\lambda_2\cos\varphi, \qquad h_{23}^{(2)} = 2\lambda_0\lambda_1\sin\varphi \\
&h_{32}^{(2)} = 2\lambda_1\lambda_2\sin\varphi.
\end{aligned} \tag{B.2}$$

The matrix $h_{\alpha\beta}^{(3)}$ is obtained from $h_{\alpha\beta}^{(2)}$ by exchanging λ_2 with λ_3. The non-vanishing components of $g_{\alpha\beta\gamma}$ are

$$\begin{aligned}
&g_{111} = -g_{122} = -g_{212} = -g_{221} = 2\lambda_0\lambda_4 \\
&g_{113} = -g_{223} = 2\lambda_0\lambda_3, \qquad g_{131} = -g_{232} = 2\lambda_0\lambda_2 \\
&g_{133} = 2\lambda_0\lambda_1\cos\varphi, \qquad g_{233} = 2\lambda_0\lambda_1\sin\varphi \\
&g_{312} = g_{321} = 2\lambda_1\lambda_4\sin\varphi, \qquad g_{311} = -2\lambda_2\lambda_3 - 2\lambda_1\lambda_4\cos\varphi \\
&g_{313} = 2\lambda_2\lambda_4 - 2\lambda_1\lambda_3\cos\varphi, \qquad g_{322} = -2\lambda_2\lambda_3 + 2\lambda_1\lambda_4\cos\varphi \\
&g_{323} = 2\lambda_1\lambda_3\sin\varphi, \qquad g_{331} = 2\lambda_3\lambda_4 - 2\lambda_1\lambda_2\cos\varphi \\
&g_{332} = 2\lambda_1\lambda_2\sin\varphi, \qquad g_{333} = \lambda_0^2 - \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_4^2.
\end{aligned} \tag{B.3}$$

PUBLICATION LIST

I. Refereed Publications

[1] **Eylee Jung**, SungHoon Kim and D. K. Park, “Newton Law on the Generalized Singular Brane with and without $4d$ Induced Gravity”, Nucl. Phys. B669 (2003) 306-324 (hep-th/0305156).

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[14] **Eylee Jung**, Mi-Ra Hwang, Hungsoo Kim, Min-Soo Kim, DaeKil Park, Jin-Woo Son and S. Tamaryan, "Reduced State Uniquely Defines the Groverian Measure of the Original Pure State", Physical Review **A 77** (2008) 062317 arXiv:0709.4292 (quant-ph): selected by APS and AIP for virtual journal of quantum information, june issue (2008): selected by APS and AIP for virtual journal of NST, june issue (2008). (2008. 06)

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II. Work in Progress and Work Submitted for Publication

[1] **Eylee Jung**, Mi-Ra Hwang, DaeKil Park and S. Tamaryan, “Three-Party Entanglement in Tripartite Teleportation Scheme through Noisy Channels”, submitted to Quantum Information and Computation, arXiv:0904.2807 (quant-ph).

국문요약

여러 부분으로 나뉜 양자상태의 얽힘에 대한 연구

물리학과 정이리
지도교수 박대길

본 학위논문에서는 pure 양자상태와 mixed 양자상태의 양자 얽힘에 대한 최근 연구결과들을 고찰하였다. 특히 여러 3-qubit 양자상태들의 기하학적 양자 얽힘 measure와 Groverian 양자 얽힘 measure를 해석적으로 구하였고, 구하여진 결과들을 local unitary 불변량으로 표현하였으며, 또한 양자상태의 parameter들로 이루어진 다각형의 기하학적양으로 재분석 하였다. 이와 같은 양자 얽힘에 대한 기하학적 해석은 앞으로 multi-party 양자상태의 양자 얽힘에 대한 깊이 있는 이해를 제공하여 줄 것으로 생각한다. 본 논문에서는 앞서 언급한 양자 얽힘의 기하학적 의미를 자세하게 논하였다. Mixed 양자상태의 경우 본 학위논문에서는 Greenberger-Horne-Zeilinger (GHZ)-, W-, 그리고 뒤집어진 W-양자상태로 이루어진 rank-3 mixture의 경우 residual entanglement 혹은 3-tangle을 해석적으로 구하였고, 이 결과를 이용하여 일반적인 Coffman-Kundu-Wootters inequality 인 monogamy inequality를 분석하였다. 또한 W-양자상태뿐 아니라 GHZ-양자상태들만으로 이루어진 특별한 mixture의 경우도 3-tangle이 그 양자 상태의 tripartite 양자 얽힘을 제대로 나타내지 못한다는 사실을 증명하였다. 이 사실은 GHZ-양자상태들로만 이루어진 특별한 mixture들은 W-상태들만으로도 optimal decomposition이 이루어진다는 것을 의미한다. 이 놀라운 결과가 의미하는 물리적 의미는 아직 확실하지 않으며 앞으로의 연구과제이다. 특히 이 사실은 Acin 등이 최근에 밝힌 "3-qubit mixed W-양자상태집합은 3-qubit 전체 양자상태 집합에서 non-zero measure를 갖는다"는 사실을 보다 물리적으로 이해하는데 실마리를 제공할 것으로 기대하고 있다.