UK INTERMEDIATE MATHEMATICAL OLYMPIAD

Hamilton Question Papers and Solutions

2008 to 2010

Organised by the

United Kingdom Mathematics Trust

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Background

The Intermediate Mathematical Olympiad and Kangaroo (IMOK) are the follow-up competitions for pupils who do extremely well in the UKMT Intermediate Mathematical Challenge (about 1 in 200 are invited to take part). The IMOK was established in 2003.

There are three written papers (Cayley, Hamilton, Maclaurin) and two multiple-choice papers (the Pink and Grey Kangaroo). The written papers each take two hours and contain six questions. Both Kangaroo papers are one hour long and contain 25 questions.

The Hamilton paper is for pupils in: Y10 or below (England and Wales); S3 or below (Scotland); School Year 11 or below (Northern Ireland).

The United Kingdom Mathematics Trust

Intermediate Mathematical Olympiad and Kangaroo (IMOK)

Olympiad Hamilton Paper

All candidates must be in *School Year 10* (England and Wales), *S3* (Scotland), or *School Year 11* (Northern Ireland).

READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, protractors and squared paper is forbidden.** Rulers and compasses may be used.
3. Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Cover Sheet on top.
4. Start each question on a fresh A4 sheet.

   You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

   Do not hand in rough work.
5. Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as π, fractions, or square roots, if appropriate, but NOT decimal approximations.
6. Give full written solutions, including mathematical reasons as to why your method is correct.

   Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.
7. **These problems are meant to be challenging!** The earlier questions tend to be easier; the last two questions are the most demanding.
   Do not hurry, but spend time working carefully on one question before attempting another.
   Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

The United Kingdom Mathematics Trust is a Registered Charity.

Enquiries should be sent to: Maths Challenges Office,
School of Mathematics, University of Leeds, Leeds, LS2 9JT.
(Tel. 0113 343 2339)
http://www.ukmt.org.uk

Advice to candidates

- *Do not hurry, but spend time working carefully on one question before attempting another.*
- *Try to finish whole questions even if you cannot do many.*
- *You will have done well if you hand in full solutions to two or more questions.*

- *Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as π, fractions, or square roots, if appropriate, but NOT decimal approximations.*

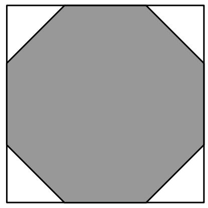
- *Give full written solutions, including mathematical reasons as to why your method is correct.*
- *Just stating an answer, even a correct one, will earn you very few marks.*
- *Incomplete or poorly presented solutions will not receive full marks.*

- ***Do not hand in rough work.***

2008

1. How many four-digit multiples of 9 consist of four different odd digits?

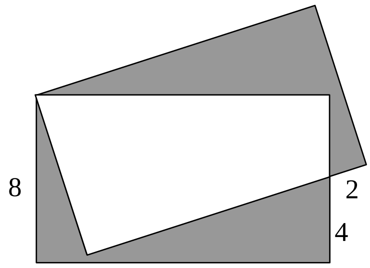
2. A regular octagon with sides of length a is inscribed in a square with sides of length 1, as shown.
   Prove that $a^2 + 2a = 1$.



3. Kelly cycles to a friend's house at an average speed of 12 km/hr. Her friend is out, so Kelly immediately returns home by the same route. At what average speed does she need to cycle home if her average speed over the whole journey is to be 15 km/hr?

4. A triangle is bounded by the lines whose equations are $y = -x - 1$, $y = 2x - 1$ and $y = k$, where k is a positive integer.
   For what values of k is the area of the triangle less than 2008?

5. Two congruent rectangles have a common vertex and overlap as shown in the diagram.
   What is the total shaded area?



6. Find all solutions to the simultaneous equations

$$x^2 - y^2 = -5$$

$$2x^2 + xy - y^2 = 5.$$

2009

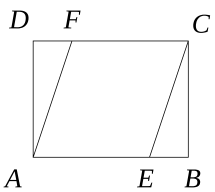
1. An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled.

   How many clownfish are in the aquarium?

2. Find the possible values of the digits p and q, given that the five-digit number '$p543q$' is a multiple of 36.

3. In the diagram, $ABCD$ is a rectangle with $AB = 16$ cm and $BC = 12$ cm. Points E and F lie on sides AB and CD so that $AECF$ is a rhombus.

   What is the length of EF?



4. Four positive integers a, b, c and d are such that:

   the sum of a and b is half the sum of c and d;

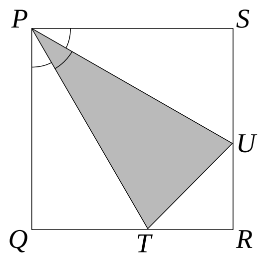
   the sum of a and c is twice the sum of b and d;

   the sum of a and d is one and a half times the sum of b and c.

   What is the smallest possible value of $a + b + c + d$?

5. The diagram shows a triangle PTU inscribed in a square $PQRS$. Each of the marked angles at P is equal to 30°.

   Prove that the area of the triangle PTU is one third of the area of the square $PQRS$.



6. Two different cuboids are placed together, face-to-face, to form a large cuboid.

   The surface area of the large cuboid is $\frac{3}{4}$ of the total surface area of the original two cuboids.

   Prove that the lengths of the edges of the large cuboid may be labelled x, y and z, where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

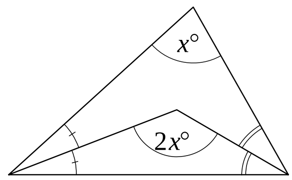
2010

1. The sum of three positive integers is 11 and the sum of the cubes of these numbers is 251.

   Find all such triples of numbers.

2. The diagram shows a triangle and two of its angle bisectors.

   What is the value of x?



3. The first and second terms of a sequence are added to make the third term. Adjacent odd-numbered terms are added to make the next even-numbered term, for example,

   first term + third term = fourth term

   and third term + fifth term = sixth term.

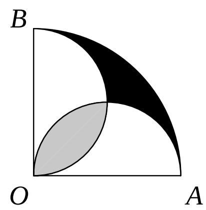
   Likewise, adjacent even-numbered terms are added to make the next odd-numbered term, for example,

   second term + fourth term = fifth term.

   Given that the seventh term equals the eighth term, what is the value of the sixth term?

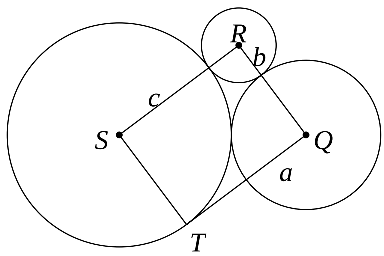
4. The diagram shows a quarter-circle with centre O and two semicircular arcs with diameters OA and OB.

   Calculate the ratio of the area of the region shaded grey to the area of the region shaded black.



5. The diagram shows three touching circles, whose radii are a, b and c, and whose centres are at the vertices Q, R and S of a rectangle $QRST$. The fourth vertex T of the rectangle lies on the circle with centre S.

   Find the ratio $a : b : c$.



6. In the diagram, the number in each cell shows the number of shaded cells with which it shares an edge or a corner. The total of all the numbers for this shading pattern is 16. Any shading pattern obtained by rotating or reflecting this one also has a total of 16.

   Prove that there are exactly two shading patterns (not counting rotations or reflections) which have a total of 17.

2	1	2
3	2	2
1	2	1

2008 Solutions

1. How many four-digit multiples of 9 consist of four different odd digits?

First solution

There are five odd digits: 1, 3, 5, 7 and 9.

The sum of the four smallest odd digits is 16 and the sum of the four largest is 24. Hence the digit sum of any four-digit number with different odd digits lies between 16 and 24, inclusive.

However, the sum of the digits of a multiple of 9 is also a multiple of 9, and the only multiple of 9 between 16 and 24 is 18. Hence the sum of the four digits is 18.

Now $1 + 3 + 5 + 9 = 18$, so that the four digits *can* be 1, 3, 5 and 9. If 7 is one of the four digits then the sum of the other three is 11, which is impossible. So 7 cannot be one of the digits and therefore the four digits can only be 1, 3, 5 and 9.

The number of arrangements of these four digits is $4 \times 3 \times 2 \times 1 = 24$. Hence there are 24 four-digit multiples of 9 that consist of four different odd digits.

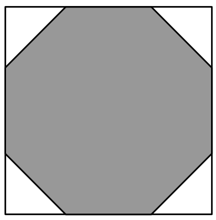
Second solution

The sum of all five odd digits is $1 + 3 + 5 + 7 + 9 = 25$.
Subtracting 1, 3, 5, 7 and 9 in turn we get 24, 22, 20, 18 and 16, only one of which is a multiple of 9, namely $18 = 25 - 7$. Since the sum of the digits of a multiple of 9 is also a multiple of 9, it follows that the four digits can only be 1, 3, 5 and 9.

The number of arrangements of these four digits is $4 \times 3 \times 2 \times 1 = 24$. Hence there are 24 four-digit multiples of 9 that consist of four different odd digits.

2. A regular octagon with sides of length a is inscribed in a square with sides of length 1, as shown.

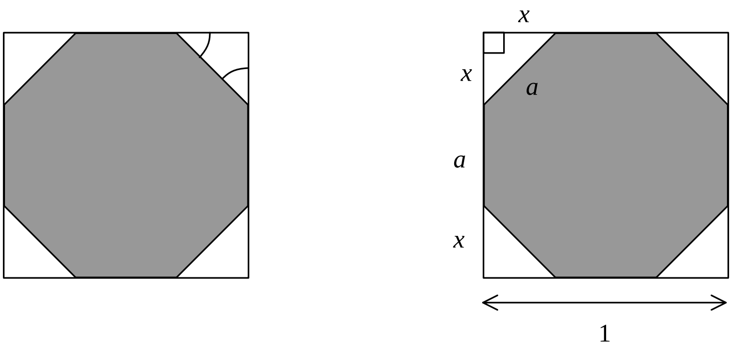
Prove that $a^2 + 2a = 1$.



First solution

Consider each of the four unshaded triangles. The angle at the vertex of a square is 90° so each triangle is right-angled.

The marked angles in the left-hand diagram are both external angles of a regular octagon, so each is equal to $\frac{1}{8} \times 360° = 45°$. Hence each triangle is isosceles (since sides opposite equal angles are equal).



Let the two equal sides of one of these triangles have length x, as shown in the right-hand diagram.

From Pythagoras' theorem $$x^2 + x^2 = a^2$$
so that $$2x^2 = a^2$$
and hence $$x = \frac{a}{\sqrt{2}}.$$

Now the side of the square has length 1, therefore
$$a + 2x = 1,$$
that is, $$a + a\sqrt{2} = 1,$$
or $$a\sqrt{2} = 1 - a.$$
Squaring this equation we get
$$2a^2 = 1 - 2a + a^2$$
and therefore $$a^2 + 2a = 1.$$

Second solution

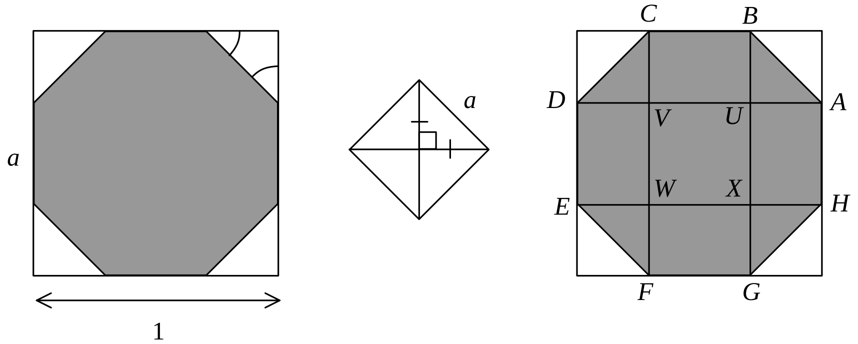
We derive an equation for a using

$$\text{unshaded area + area of octagon = area of the square.} \qquad (1)$$

Consider each of the four unshaded triangles. The angle at the vertex of a square is 90° so each triangle is right-angled.

The marked angles in the left-hand diagram are both external angles of a regular octagon, so each is equal to $\frac{1}{8} \times 360° = 45°$. Hence each triangle is isosceles (since sides opposite equal angles are equal).

Therefore each of the four unshaded triangles is isosceles and right-angled, with hypotenuse of length a, so the four triangles can be reassembled to form a square of side a (see below). Hence the unshaded area is equal to a^2.



Similarly, the four shaded triangles in the right-hand figure together have an area of a^2.

The octagon comprises these four shaded triangles together with two rectangles, $ADEH$ and $BCFG$, which overlap in the square $UVWX$. Therefore the area of the octagon is

$$a^2 + \text{area}\, ADEH + \text{area}\, BCFG - \text{area}\, UVWX.$$

But the two rectangles each have area $a \times 1$ and the area of square $UVWX$ is $a \times a$, so that the octagon has area

$$a^2 + a + a - a^2 = 2a.$$

Finally, the large square has area 1, so equation (1) gives

$$a^2 + 2a = 1.$$

3. Kelly cycles to a friend's house at an average speed of 12 km/hr. Her friend is out, so Kelly immediately returns home by the same route. At what average speed does she need to cycle home if her average speed over the whole journey is to be 15 km/hr?

First solution

Let the distance cycled to the house be d km; let the time taken on the journey to the friend's house be t_1 hours and let the time taken on the way back be t_2 hours.
From the given information about average speeds,

$$12 = \frac{d}{t_1}$$

and

$$15 = \frac{2d}{t_1 + t_2}.$$

These equations may be rearranged to give

$$12t_1 = d \qquad (1)$$

and

$$15t_1 + 15t_2 = 2d. \qquad (2)$$

Substituting from equation (1) into equation (2), we get

$$15t_1 + 15t_2 = 24t_1$$

so that

$$t_1 = \tfrac{5}{3}t_2.$$

Then equation (1) gives

$$12 \times \tfrac{5}{3}t_2 = d,$$

and hence

$$20 = \frac{d}{t_2}.$$

Thus Kelly's average speed cycling home is 20 km/h.

Second solution

Let the distance cycled to the house be d km; let the average speed for the journey home be v km/h. Then from the information given

the time for the outward journey $= \dfrac{d}{12}$ hours,

the time for the homeward journey $= \dfrac{d}{v}$ hours,

and the time for the whole journey $= \dfrac{2d}{15}$ hours.

Therefore we have

$$\frac{d}{12} + \frac{d}{v} = \frac{2d}{15},$$

which may be rearranged to give

$$\frac{1}{v} = \frac{2}{15} - \frac{1}{12} = \frac{8 - 5}{60} = \frac{1}{20}.$$

Hence $v = 20$ and Kelly's average speed cycling home is 20 km/h.

4. A triangle is bounded by the lines whose equations are $y = -x - 1$, $y = 2x - 1$ and $y = k$, where k is a positive integer.

For what values of k is the area of the triangle less than 2008?

Solution

The lines with equations $y = -x - 1$ and $y = 2x - 1$ intersect when

$$-x - 1 = 2x - 1,$$

from which $$x = 0,$$

so that the lines meet at $(0, -1)$.

The line $y = k$ intersects the line $y = -x - 1$ when

$$k = -x - 1,$$

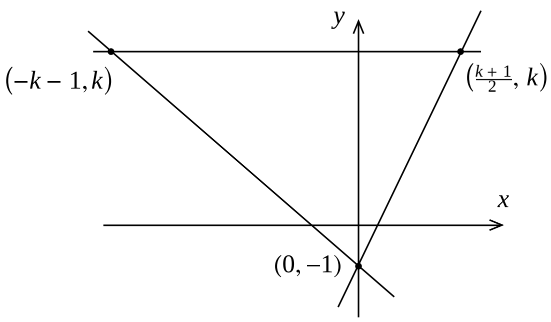
from which $$x = -k - 1,$$

and the line $y = k$ intersects the line $y = 2x - 1$ when

$$k = 2x - 1$$

from which $$x = \frac{k+1}{2}.$$

Thus the three intersection points are $(0, -1)$, $(-k - 1, k)$ and $\left(\frac{k+1}{2}, k\right)$.



Now the enclosed triangle has height $k + 1$ and 'base' equal to

$$\frac{k+1}{2} - (-k - 1) = \frac{k+1}{2} + k + 1$$
$$= \tfrac{3}{2}(k + 1),$$

so the enclosed area is

$$\tfrac{1}{2} \times \tfrac{3}{2}(k + 1) \times (k + 1) = \tfrac{3}{4}(k + 1)^2.$$

Therefore when the area is less than 2008,

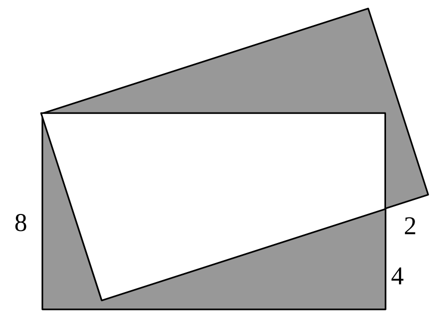
$$\tfrac{3}{4}(k + 1)^2 < 2008,$$

so that $$(k + 1)^2 < \frac{8032}{3}$$
$$= 2677\tfrac{1}{3}.$$

Now $51^2 = 2601$ and $52^2 = 2704$ so that $k + 1 < 52$, that is, $k < 51$. Hence the possible values of k are given by $1 \leqslant k \leqslant 50$.

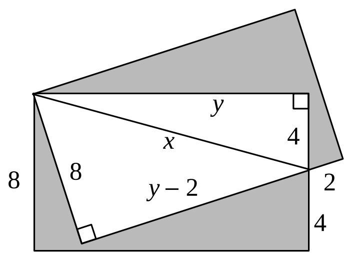
5. Two congruent rectangles have a common vertex and overlap as shown in the diagram.

What is the total shaded area?



Solution

Let the unknown side of the rectangle have length y and let one diagonal of the unshaded quadrilateral have length x, as shown below.



Applying Pythagoras' theorem to the two unshaded right-angled triangles we get

$$x^2 = 8^2 + (y - 2)^2$$

and $$x^2 = y^2 + 4^2.$$

Eliminate x^2 from these equations to give

$$64 + (y - 2)^2 = y^2 + 16,$$

that is, $$64 + y^2 - 4y + 4 = y^2 + 16,$$

which rearranges to $$52 = 4y$$

and hence $$y = 13.$$

Now the shaded area is equal to twice the area of one rectangle minus twice the area of the unshaded region, that is,

$$2(8 \times 13) - 2\left(\tfrac{1}{2} \times 8 \times 11 + \tfrac{1}{2} \times 4 \times 13\right) = 68.$$

Hence the total shaded area equals 68.

6. Find all solutions to the simultaneous equations

$$x^2 - y^2 = -5$$

$$2x^2 + xy - y^2 = 5.$$

First solution

We may rewrite the given equations by factorising the left-hand sides:

$$(x - y)(x + y) = -5 \qquad (1)$$

$$(2x - y)(x + y) = 5. \qquad (2)$$

Since -5 is non-zero, we may divide (2) by (1) to get

$$\frac{2x - y}{x - y} = -1,$$

which rearranges to $$2x - y = y - x$$

and hence $$x = \tfrac{2}{3}y.$$

Now substitute $x = \tfrac{2}{3}y$ in $x^2 - y^2 = -5$ to obtain

$$\tfrac{4}{9}y^2 - y^2 = -5,$$

so that $$y^2 = 9$$

and hence $$y = \pm 3.$$

Since $x = \tfrac{2}{3}y$ we deduce that the equations have two solutions:

$$x = 2, y = 3 \qquad \text{and} \qquad x = -2, y = -3.$$

Second solution

The given equations are

$$x^2 - y^2 = -5 \tag{3}$$

$$2x^2 + xy - y^2 = 5. \tag{4}$$

Adding (3) and (4) we get

$$3x^2 + xy - 2y^2 = 0$$

which factorises to $$(3x - 2y)(x + y) = 0.$$

Hence $x = \tfrac{2}{3}y$ or $x = -y$. But, from equation (3), we know $x \neq -y$ so we have $x = \tfrac{2}{3}y$.

Substitute $x = \tfrac{2}{3}y$ in (3) to obtain

$$\tfrac{4}{9}y^2 - y^2 = -5,$$

so that $$y^2 = 9$$

and hence $$y = \pm 3.$$

Since $x = \tfrac{2}{3}y$ we deduce that the equations have two solutions:

$$x = 2, y = 3 \qquad \text{and} \qquad x = -2, y = -3.$$

2009 Solutions

1. An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled.

   How many clownfish are in the aquarium?

   Solution
   Let there be x clownfish in the aquarium.
   If 60 clownfish are added there are $x + 60$ clownfish and 340 tropical fish in total.
   Since the proportion of clownfish is then doubled, we have

$$2 \times \frac{x}{280} = \frac{x + 60}{340}.$$

   Multiplying both sides by 20, we get

$$\frac{x}{7} = \frac{x + 60}{17}$$

   and hence

$$17x = 7(x + 60).$$

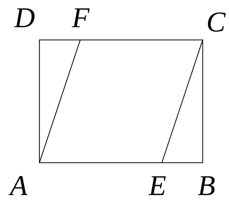
   It follows that $x = 42$ and thus there are 42 clownfish in the aquarium.

2. Find the possible values of the digits p and q, given that the five-digit number '$p543q$' is a multiple of 36.

   Solution
   Since '$p543q$' is a multiple of 36 it is a multiple of both 9 and 4.
   The sum of the digits of a multiple of 9 is also a multiple of 9, hence $p + 5 + 4 + 3 + q$ is a multiple of 9. But $5 + 4 + 3 = 12$ and each of p and q is a single digit, so that $p + q = 6$ and $p + q = 15$ are the only possibilities.
   Since '$p543q$' is a multiple of 4 and '$p5400$' is always divisible by 4, it follows that '$3q$' is divisible by 4. The only possible values for '$3q$' are 32 and 36, so that $q = 2$ or $q = 6$.
   If $q = 2$, then $p + q = 15$ is not possible since p is a single digit. Hence $p + q = 6$ and so $p = 4$.
   If $q = 6$, then $p + q = 6$ is not possible since '$p543q$' is a five-digit number and therefore the digit p cannot be zero. Hence $p + q = 15$ and so $p = 9$.
   Therefore $p = 4$, $q = 2$ and $p = 9$, $q = 6$ are the only possible values of the digits p and q.

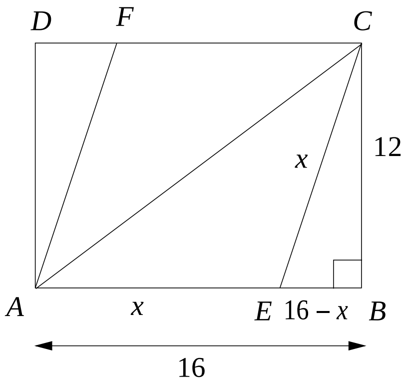
3. In the diagram, $ABCD$ is a rectangle with $AB = 16$ cm and $BC = 12$ cm. Points E and F lie on sides AB and CD so that $AECF$ is a rhombus.

What is the length of EF?



Solution

Let the sides of the rhombus $AECF$ have length x cm. Hence $AE = x$ and $EB = 16 - x$. Since $ABCD$ is a rectangle, angle EBC is a right angle.



Using Pythagoras' theorem in triangle ABC, we have $AC^2 = 16^2 + 12^2 = 400$, so that $AC = 20$ cm.

Using Pythagoras' theorem in triangle EBC, we have

$$EC^2 = CB^2 + EB^2$$

and hence

$$x^2 = 12^2 + (16 - x)^2,$$

which can be rearranged to give

$$x^2 = 144 + 256 - 32x + x^2.$$

It follows that

$$32x = 400$$

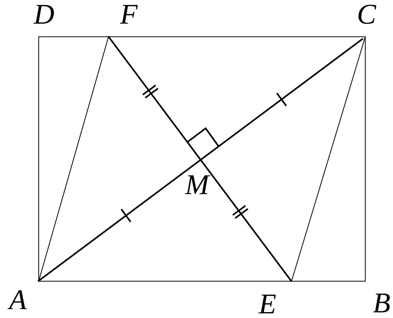
and so

$$x = \frac{25}{2}.$$

We may now proceed in various ways; we show two different methods.

First method

Let M be the point of intersection of the diagonals AC and EF of $AECF$. Since $AECF$ is a rhombus, angle FMC is a right angle and M is the mid-point of both AC and EF.



Using Pythagoras' theorem in triangle *FMC*, we have

$$CF^2 = \left(\tfrac{1}{2}AC\right)^2 + \left(\tfrac{1}{2}EF\right)^2$$

and hence

$$\left(\frac{25}{2}\right)^2 = 10^2 + \left(\tfrac{1}{2}EF\right)^2.$$

It follows that

$$625 = 400 + EF^2$$

and so

$$EF = 15.$$

Therefore the length of *EF* is 15 cm.

Second method

We make use of the fact that

area of rhombus *AECF* = area of rectangle *ABCD* $-$ 2 $\times$ area of triangle *EBC*.

Now the area of a rhombus is half the product of its diagonals. Also, the area of triangle *EBC* is $\tfrac{1}{2}EB \times BC$ and $EB = 16 - \tfrac{25}{2} = \tfrac{7}{2}$. We therefore have

$$\tfrac{1}{2}AC \times EF = 16 \times 12 - \tfrac{7}{2} \times 12.$$

Hence

$$10 \times EF = 192 - 42 = 150$$

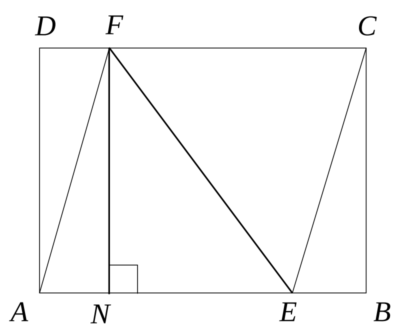
and so

$$EF = 15.$$

Therefore the length of *EF* is 15 cm.

Remark

Another method uses Pythagoras' theorem in the right-angled triangle *ENF* shown below.



4. Four positive integers a, b, c and d are such that:

the sum of a and b is half the sum of c and d;

the sum of a and c is twice the sum of b and d;

the sum of a and d is one and a half times the sum of b and c.

What is the smallest possible value of $a + b + c + d$?

Solution

There are three equations here but four unknown values, a, b, c and d. Thus it is not possible just to solve the equations to find the values of a, b, c and d. What we can do is to find relationships between them and then deduce possible values of a, b, c and d.

From the given information,

$$a + b = \tfrac{1}{2}(c + d) \qquad (1)$$

$$a + c = 2(b + d) \qquad (2)$$

$$a + d = \tfrac{3}{2}(b + c). \qquad (3)$$

We may proceed in various ways; we show two methods, substitution and elimination.

First method: *substitution*

From (1), we have

$$a = -b + \tfrac{1}{2}(c + d). \qquad (4)$$

Substituting in (2), we get

$$-b + \tfrac{1}{2}(c + d) + c = 2(b + d)$$

and hence

$$\tfrac{3}{2}c - \tfrac{3}{2}d = 3b,$$

that is,

$$c - d = 2b. \qquad (5)$$

Substituting from (4) in (3), we get

$$-b + \tfrac{1}{2}(c + d) + d = \tfrac{3}{2}(b + c)$$

so that

$$-c + \tfrac{3}{2}d = \tfrac{5}{2}b. \qquad (6)$$

Now adding (5) and (6) we obtain

$$\tfrac{1}{2}d = \tfrac{9}{2}b,$$

and hence

$$d = 9b.$$

Once we have minimised $b + d$, then we automatically minimise $a + c$, because of equation (2), and hence minimise the sum we are interested in.

Since b and d are positive integers, $b = 1$ and $d = 9$ are the smallest possible values with $d = 9b$. From (5) and (4), we see that the corresponding values of c and a are $c = 11$ and $a = 9$, both of which are also positive integers, as required.

Checking these values in equations (1) to (3), we confirm that they are valid solutions of the given equations.

Hence the smallest possible value of $a + b + c + d$ is 30.

Second method: *elimination*

We may rearrange the three equations (1), (2) and (3) to give

$$2a + 2b = c + d \tag{7}$$

$$a + c = 2b + 2d \tag{8}$$

$$2a + 2d = 3b + 3c. \tag{9}$$

Adding (7) and (8), we get

$$3a + 2b + c = 2b + c + 3d$$

and hence

$$a = d.$$

Then (7) and (9) may be rewritten

$$2b - c + d = 0. \tag{10}$$

and

$$3b + 3c - 4d = 0. \tag{11}$$

Now adding $3 \times$ (10) and (11), we obtain

$$9b - d = 0$$

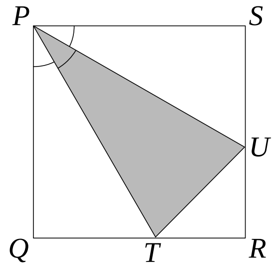
and hence

$$d = 9b.$$

The solution now proceeds in the same way as the first method.

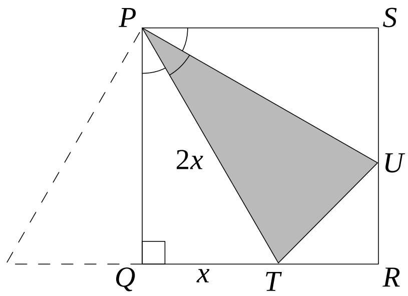
5. The diagram shows a triangle PTU inscribed in a square $PQRS$. Each of the marked angles at P is equal to $30°$.

Prove that the area of the triangle PTU is one third of the area of the square $PQRS$.



First solution

Let $QT = x$, so that $PT = 2x$, since triangle PTQ is half an equilateral triangle.



Using Pythagoras' theorem in triangle PTQ, we get

$$\begin{aligned} PQ^2 &= PT^2 - QT^2 \\ &= (2x)^2 - x^2 \\ &= 3x^2 \end{aligned}$$

and hence

$$PQ = \sqrt{3}x.$$

We can now find the areas of the three unshaded right-angled triangles.

$$\begin{aligned} \text{Area of triangle } PQT &= \tfrac{1}{2} \times x \times \sqrt{3}x \\ &= \frac{\sqrt{3}}{2}x^2. \end{aligned}$$

Similarly,

$$\text{area of triangle } PSU = \frac{\sqrt{3}}{2}x^2.$$

Finally,

$$\begin{aligned} \text{area of triangle } TRU &= \tfrac{1}{2} \times \left(\sqrt{3}x - x\right) \times \left(\sqrt{3}x - x\right) \\ &= \tfrac{1}{2}\left(3x^2 - 2\sqrt{3}x^2 + x^2\right) \\ &= 2x^2 - \sqrt{3}x^2. \end{aligned}$$

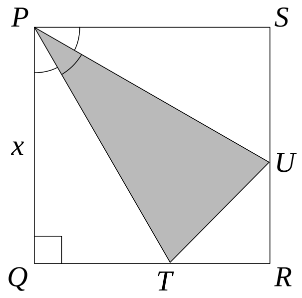
Therefore the total unshaded area is

$$\frac{\sqrt{3}}{2}x^2 + \frac{\sqrt{3}}{2}x^2 + 2x^2 - \sqrt{3}x^2 = 2x^2.$$

However, the area of the square $PQRS$ is $\left(\sqrt{3}x\right)^2 = 3x^2$. It follows that the shaded area is x^2, which is one third of the area of the square.

Second solution

Let the sides of the square $PQRS$ have length x.



Then in triangle PQT we have

$$\cos 30^\circ = \frac{x}{PT},$$

and hence

$$PT = \frac{x}{\cos 30^\circ}.$$

Now by symmetry $PU = PT$ so that

$$\begin{aligned}\text{area of triangle } PTU &= \tfrac{1}{2}PT \times PU \sin \angle TPU \\ &= \tfrac{1}{2}x^2 \frac{\sin 30^\circ}{\cos^2 30^\circ}.\end{aligned}$$

Now $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = \frac{1}{2}$. Therefore

$$\begin{aligned}\text{area of triangle } PTU &= \tfrac{1}{2}x^2 \times \frac{\frac{1}{2}}{\frac{3}{4}} \\ &= \tfrac{1}{3}x^2.\end{aligned}$$

Hence the area of the triangle PTU is one third of the area of the square $PQRS$.

6. Two different cuboids are placed together, face-to-face, to form a large cuboid.

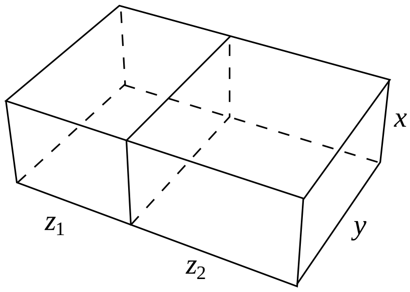
The surface area of the large cuboid is $\frac{3}{4}$ of the total surface area of the original two cuboids.

Prove that the lengths of the edges of the large cuboid may be labelled x, y and z, where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

First solution

Since the two cuboids are placed together, face-to-face, to form a large cuboid, two of the edges have the same lengths. Let these common lengths be x and y, and let the other edges of the two cuboids have lengths z_1 and z_2, as shown.



Now the surface area of the large cuboid is $\frac{3}{4}$ of the total surface area of the original two cuboids. Therefore

$$2\left[xy + x(z_1 + z_2) + y(z_1 + z_2)\right] = \tfrac{3}{4}\left[2(xy + z_1x + z_1y) + 2(xy + z_2x + z_2y)\right]$$

so that

$$4\left[xy + x(z_1 + z_2) + y(z_1 + z_2)\right] = 3\left[2xy + x(z_1 + z_2) + y(z_1 + z_2)\right]$$

and hence

$$2xy = x(z_1 + z_2) + y(z_1 + z_2),$$

that is,

$$2xy = (x + y)(z_1 + z_2).$$

Now $z_1 + z_2 = z$, where z is the edge length of the large cuboid. Therefore

$$2xy = (x + y)z$$

which may be rearranged to give

$$\frac{2}{z} = \frac{x + y}{xy}$$

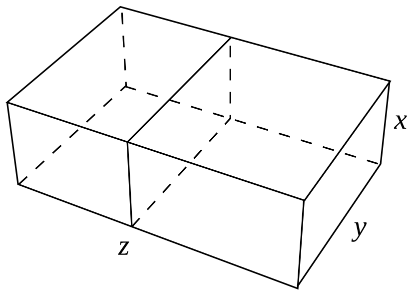
$$= \frac{1}{y} + \frac{1}{x}.$$

Hence the lengths of the edges of the large cuboid may be labelled x, y and z, where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

Second solution

Let the large cuboid have dimensions x, y and z, as shown.



Now the total surface area T of the two original cuboids is equal to the surface area of the large cuboid added to the area of the two faces which are joined together. But the surface area of the large cuboid is $\frac{3}{4}T$, hence the area of the two faces which are joined together is $\frac{1}{4}T$, that is, $\frac{1}{3}$ of the surface area of the large cuboid.

Therefore

$$2xy = \tfrac{1}{3}(2xy + 2yz + 2zx)$$

so that

$$6xy = 2xy + 2yz + 2zx$$

and hence

$$2xy = yz + zx.$$

Dividing by xyz, we obtain, as required,

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

2010 Solutions

1 The sum of three positive integers is 11 and the sum of the cubes of these numbers is 251.

Find all such triples of numbers.

Solution
Let us calculate the first few cubes in order to see what the possibilities are:

$$1^3 = 1,\ 2^3 = 8,\ 3^3 = 27,\ 4^3 = 64,\ 5^3 = 125,\ 6^3 = 216 \text{ and } 7^3 = 343. \quad (*)$$

The sum of the cubes of the positive integers is 251, which is less than 343, hence none of the integers is greater than 6.

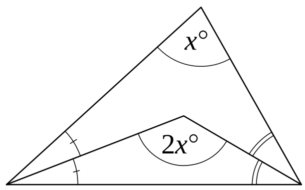
Now $\frac{251}{3} = 83\frac{2}{3} > 64 = 4^3$, therefore at least one of the integers is 5 or more.

If one of the integers is 6, then the other two cubes add up to $251 - 6^3 = 251 - 216 = 35$. From (*) above, $3^3 + 2^3 = 27 + 8 = 35$ is the only possibility. Also, $6 + 3 + 2 = 11$ so that 6, 3 and 2 is a possible triple of numbers.

If one of the integers is 5, then the other two cubes add up to $251 - 5^3 = 251 - 125 = 126$. From (*) above $5^3 + 1^3 = 125 + 1 = 126$ is the only possibility. Also, $5 + 5 + 1 = 11$ so that 5, 5 and 1 is a possible triple of numbers.

Hence 2, 3, 6 and 1, 5, 5 are the triples of numbers satisfying the given conditions.

2 The diagram shows a triangle and two of its angle bisectors.
What is the value of x?



Solution
Let the sum of the two unlabelled angles in the smaller triangle be y. Then the sum of the two unlabelled angles in the whole triangle is equal to $2y$.

The sum of the angles in a triangle is 180°, hence in the small triangle

$$2x + y = 180 \tag{2.1}$$

and in the whole triangle

$$x + 2y = 180. \tag{2.2}$$

Doubling equation (2.1) and subtracting equation (2.2), we get $3x = 180$ and thus $x = 60$.

3 The first and second terms of a sequence are added to make the third term. Adjacent odd-numbered terms are added to make the next even-numbered term, for example,

first term + third term = fourth term

and third term + fifth term = sixth term.

Likewise, adjacent even-numbered terms are added to make the next odd-numbered term, for example,

second term + fourth term = fifth term.

Given that the seventh term equals the eighth term, what is the value of the sixth term?

Solution

Let a be the first term of the sequence and b the second term. Thus the first eight terms of the sequence are:

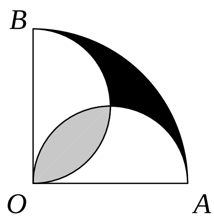
$$a,\ b,\ a + b,\ 2a + b,\ 2a + 2b,\ 3a + 3b,\ 5a + 4b,\ 7a + 6b.$$

The seventh term equals the eighth term, hence $5a + 4b = 7a + 6b$. Therefore $2a + 2b = 0$ and so $a = -b$.

Hence the value of the sixth term is $3a + 3b = -3b + 3b = 0$.

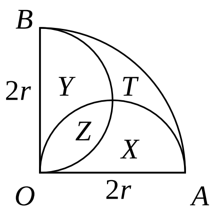
4 The diagram shows a quarter-circle with centre O and two semicircular arcs with diameters OA and OB.

Calculate the ratio of the area of the region shaded grey to the area of the region shaded black.



Solution

Let $2r$ be the radius of the quarter-circle. Hence the radius of each semicircle is r. The diagram is divided into four regions; let their areas be X, Y, Z and T, as shown below.



The area of the quarter-circle is $\frac{1}{4}\pi\,(2r)^2 = \pi r^2$. The area of each semicircle is $\frac{1}{2}\pi r^2$. Hence $X + Z = \frac{1}{2}\pi r^2$.

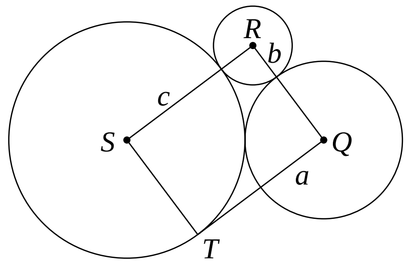
However, the area inside the quarter-circle but outside one semicircle is $\pi r^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi r^2$. This means that $X + T = \frac{1}{2}\pi r^2$.

Therefore $X + T = X + Z$. We conclude that $T = Z$, so that the areas of the shaded regions are equal.

Thus the ratio of the area of the region shaded grey to the area of the region shaded black is $1 : 1$.

5 The diagram shows three touching circles, whose radii are a, b and c, and whose centres are at the vertices Q, R and S of a rectangle $QRST$. The fourth vertex T of the rectangle lies on the circle with centre S.

Find the ratio $a : b : c$.



Solution

In the rectangle $QRST$, we have $QR = TS$ and hence

$$a + b = c. \tag{5.1}$$

In the right-angled triangle QRS, by Pythagoras' Theorem, $QS^2 = QR^2 + RS^2$. But $QS = a + c$, $QR = a + b$ and $RS = b + c$, therefore

$$(a + c)^2 = (a + b)^2 + (b + c)^2. \tag{5.2}$$

Substituting for a from equation (5.1) into equation (5.2), we get

$$(2c - b)^2 = c^2 + (b + c)^2.$$

Thus

$$4c^2 - 4bc + b^2 = c^2 + b^2 + 2bc + c^2,$$

so that

$$2c^2 - 6bc = 0$$

and hence

$$c\,(c - 3b) = 0.$$

But $c \neq 0$, hence $c = 3b$. Again from equation (5.1), $a + b = 3b$ and thus $a = 2b$.

Therefore the ratio $a : b : c = 2 : 1 : 3$.

6 In the diagram, the number in each cell shows the number of shaded cells with which it shares an edge or a corner. The total of all the numbers for this shading pattern is 16. Any shading pattern obtained by rotating or reflecting this one also has a total of 16.

2	1	2
3	2	2
1	2	1

Prove that there are exactly two shading patterns (not counting rotations or reflections) which have a total of 17.

Solution

Whenever a cell is shaded, one is added to all the cells with which it shares an edge or corner. So consider an alternative numbering system: in each shaded cell write the number of cells with which it shares an edge or corner; leave each unshaded cell blank. For example, for the shading pattern given in the question we obtain:

	5	
	8	
3		

This is equivalent to the original numbering system; in particular, the total of all the numbers is the same.

Now a shaded corner cell has 3 adjacent cells; a shaded edge cell has 5 adjacent cells; the shaded central cell has 8 adjacent cells. Thus the total of all the numbers for a shading pattern is made up solely by adding multiples of 3, 5 and 8.

For a 3×3 diagram the available numbers are therefore: four 3s, four 5s and one 8.

If the 8 is used, a remaining total of $17 - 8 = 9$ is required. The only way to attain 9 is to use three 3s.

If the 8 is not used, since 17 is not a multiple of 3 at least one 5 is needed. Now $17 - 1 \times 5 = 12$, $17 - 2 \times 5 = 7$ and $17 - 3 \times 5 = 2$, but neither 7 nor 2 is a multiple of 3. So the only possibility is to use one 5 and then a remaining total of 12 is required. The only way to attain 12 is to use four 3s.

Thus the only possibilities are: 3, 3, 3, 3, 5 and 3, 3, 3, 8. Both of these are possible using the available numbers. What are the corresponding shading patterns?

3	5	3
3		3

3		3
	8	
3		

The diagrams above give examples of the only possible shading pattern for each set of numbers—all others are rotations of one of these. In the first case, the four corners are shaded to obtain four 3s, then there is only one way, up to rotation, to shade an edge cell to obtain the 5. In the second case, the centre is shaded to obtain the 8, then there is only one way, up to rotation, to shade three corner cells to obtain three 3s.

Therefore there are exactly two shading patterns with a total of 17.